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Beyond circular trigonometry: Parabolic functions from geometric identities

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***Abstract*:**
This paper presents an innovative extension of trigonometric functions to parabolic geometry, introducing the parabolic sine (sinp u) and parabolic cosine (cosp u) functions. Geometrically, sinp u and cosp u are defined via the relationship between a point on a parabola and its focus: sinp u represents the vertical displacement ratio, while cosp u corresponds to the horizontal displacement ratio, normalized by the focal distance. These functions generalize circular trigonometry to a parabolic framework, preserving key structural identities while exhibiting unique behaviors, such as fixed asymptotic values under angle variation. The objective of this study is to establish a rigorous foundation for parabolic trigonometry, derive its core identities, and demonstrate its applicability. Using a geometric-analytic approach, we redefine trigonometric concepts via parabola-centric constructions, adapt Euler's formula to parabolic segments, and derive exponential representations of sinp u and cosp u. This method leverages differential geometry and algebraic invariance to ensure consistency with classical trigonometry while extending its scope. Key results include: (1) Proofs of sinp u, and cosp u; (2) Exponential forms: sinp u, and cosp u; (3) As the parabolic imaginary unit. Unlike circular trigonometry adaptations, our approach provides intrinsic geometric consistency with parabolic functions, enabling exact solutions for parabolic arc lengths and focal properties. This contrasts with numerical or linearized methods that sacrifice accuracy for simplicity. Theoretically, unifies parabolic geometry with analytic trigonometry, opening pathways for conic-section-generalized trigonometry, enhancing modeling in optics (parabolic mirrors), structural engineering (cable-supported arches), and ballistics (trajectory optimization), offering a novel pedagogical tool to bridge classical and modern geometry.

Keywords: Applied Geometry; Conic sections; Euler's formula; Geometric identities; Parabolic trigonometry.

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Introduction

At first glance, trigonometric functions seem tied to circles and hyperbolas. The key difference between classical trigonometric functions (*circle and hyperbolic*) is that classical (*circular*) trigonometric functions are based on the unit circle $x^2 + y^2 = 1$, where the circle represents periodic motion (*e.g., rotations, oscillations*), and these functions periodic and describe rotational symmetr y (Al-Ossmi, 2024; Jeffrey & Dai, 2008; Vallo et al., 2022), satisfy the Pythagorean identity. Arise naturally in problems involving waves, circular motion, and oscillations. While Hyperbolic trigonometric functions are based on the unit hyperbola x^2 - y^2 = 1, the hyperbola is related to exponential growth/decay and non-periodic phenomena. For a "*hyperbolic angle*" t, the coordinates (x, y) on the hyperbola, and these satisfy the hyperbolic identity (Dattoli et al., 2011). These functions are not periodic (grow exponentially), describe hyperbolic symmetry, and they arise in relativity, catenary curves, and exponential processes (Dattoli et al., 2011; Nielsen et al., 2017). Hyperbolic trig functions solve problems with exponential growth and Lorentz transformations (Al-ossmi, 2023; Charkaoui & Alaa, 2022), while circular trig functions solve problems with rotations and periodicity. Just as circular and hyperbolic functions correspond to circles and hyperbolas, parabolic functions correspond to parabolas (y^2 = x).

While circular and hyperbolic trig functions are well-known, parabolic trigonometry is a newer and less standardized field. From the classification of second-order ODEs, Parabolic functions are parabolic degenerate (Chemin, 2005; Dattoli et al., 2011). This reflects the fact that parabolic symmetry is degenerate (no oscillation or exponential growth, just linear scaling). In nilpotent geometry (a branch of differential geometry), parabolic functions describe shear transformations (like $x \to x + y$, $y \to y$), here the "parabolic angle" t parameterizes shear flow (Dattoli et al., 2011), also, the functions are polynomial. These appear in control theory and non-Euclidean geometry. The parabola is the boundary case between ellipses (closed orbits), and hyperbolas, open trajectories (Kuttler, 2007; Lyachek, 2020; Menzler-Trott, 2007). Also, in matters of conic sections, there are other properties such that it helps to group the circle and hyperbola in one, and the parabola and ellipse in the other. In the Euclidean geometry, it has curvature is K=0, so the parabolic sine and the parabolic cosine would be the functions which make the similar formulas true in Euclidean geometry. In this case, there is a "natural" choice for the circular functions, and a "natural" choice for the hyperbolic functions. But there is no 'natural' choice for the ellipse (in fact, a circle is a kind of ellipse, so the circular functions "are" the 'natural' case of the elliptical ones). The hyperbola x^2 - y^2 = 1 is parametrized by *cosh*,and *sinh* (Cannone, 2005; Nielsen et al., 2017). If you're willing to go one degree beyond quadratics, nonsingular cubics can be transformed into elliptic curves, which can be parametrized with Weierstrass $\wp$-functions and their first derivatives (Larson et al., 2013; Vodop'yanov & Kudryavtseva, 2019). With no other mission but to parameterize $y = x^2$, it could easily take "parabolic cosine" to be a fairly exotic bijection, not necessarily the identity; then "parabolic sine" would be the square *of that function*, not necessarily the square function itself, according to conic definition, this corresponds to the fact that if you view conic sections as being given by slicing a cone with a plane, you obtain the parabola in the boundary between ellipses and hyperbolas.

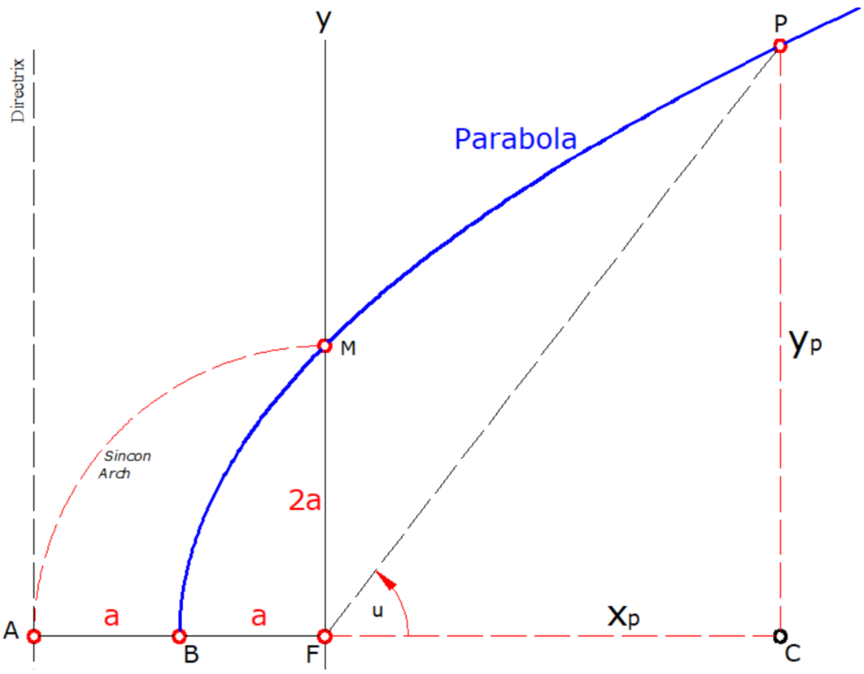
It is not at all unreasonable to seek "parabolic" fucnctions "not-rectangularly-hyperbolic" variants of these circular trig functions. In geometry, parabolic trigonometric functions describes scaling laws fractals, shear flows, and nilpotent dynamics (Lyachek, 2020; Vodopyanov, 2019). The answer lies in symmetry, differential equations, and completing the "trigonometric trilogy. In physics, parabolic motion (e.g., projectiles) appears when a system is critically damped (neither oscillating nor purely exponential). In differential equations, parabolic PDEs (e.g., the heat equation) describe diffusion, unlike wave equations (hyperbolic), or Laplace's equation elliptic (Cannone, 2005; Grinshpan, 2010). Thus, parabolic trigonometric functions fill the missing link in the classification of symmetries. These functions describe shear transformations which appear in control theory (optimal trajectories) and non-Euclidean geometry (nilpotent groups), since parabolic curves appear in optimal control (e.g., time-optimal trajectories), critical damping in engineering (e.g., suspension systems), Models with memory effects (e.g., fracture mechanics, finance), and time-optimal paths often follow parabolic scaling laws (Al-ossmi, 2023; Azhary Masta et al., 2018; Novruzi, 2023; Papageorgiou et al., 2019). In Mathmatic , a parabola is a curve where any point is at an equal distance from a fixed point (the focus), and a fixed straight line (the directrix) (Novruzi, 2023; Faraoni, 2013;Volenec *et al.,* 2021). The vertex (where the parabola makes its sharpest turn) is halfway between the focus and directrix. The equation for a parabola is generally; $y^2 = 4ax$, where a is the distance from the origin to the focus, (and also from the origin to directrix) (Nielsen et al., 2017; Spíchal, 2022). The curves can also be defined using a straight line and a point (called the directrix and focus). The latus rectum runs parallel to the directrix and passes through the focus. For a parabola segment whose axis is the x-axis and with vertex at the origin, the equation in which; $a > 0$ is the distance between the directrix and the focus (Zarco & Pascual-Fuentes, 2023), the parabolic functions in this paper take a real argument called a parabolic angle (u).

Introducing trigonometric functions based on the parabola provides a powerful framework for addressing geometric problems that involve parabolic trajectories, optimization of reflective surfaces, and the analysis of light or sound paths(Grinshpan, 2010). These functions offer alternative ratios and identities suited for parabolic curves, enriching the mathematician's and engineer's toolbox, especially in fields like physics, architecture, and astronomy where parabolas frequently occur. There is a lot of literature concerning equations interacting with a parabolic curvature (Dattoli et al., 2011; Stewart, 2016). Some specific facts in this field were applied with elementary knowledge of functional analysis elliptic and parabolic equations have important application in the field of partial differential equations (Chemin, 2005). parabolic equations also used in the theory of Sobolev's spaces (Vodopyanov, 2019), and Holder's spaces (Kuttler, 2007), Minkowski's spaces (Faraoni, 2013), and Young's inequality (Cannone, 2005), and recently (Sarfraz et al., 2021).

Just as complex numbers unified algebra, parabolic trigonometry may helps unify geometric symmetries and differential equations. In this case, there have been a variety of applications to this purpose, however, an exhaustive theory of these kinds of equations is outside the scope of this paper. In this paper, the parabola trigonometric functions based on geometrical identities and emerged as an extension of traditional trigonometry, redefining angle-based relationships according to the geometric properties of the parabola rather than the circle. In this paper, parabolic functions based on geometry of unit circle are non-periodic trigonometric functions, but with power-law behavior (*not exponential*). Analogous to how *sin u* and *cos u* parameterize their respective curves, parabolic trigonometric functions (*sinp u*, *cosp u*, and *tanp u*) parameterize the parabola, since they describe parabolic scaling symmetry, they are designed to be used in problems involving fractional calculus, dynamical systems, and certain differential equations.

Research Methods

This research is a pure mathematical research based on theory, in this paper, a geometric approach is adopted to investigate the relationship between trigonometric functions based on propertions of the parabolic segment curve (Ammad et al., 2022), aiming to derive new functions intrinsically linked to the parabola's geometric properties. The methodology involves positioning the parabola so that its focus coincides with the origin. We simply generalize the centered unit circle to a conic segment of parabola with focus at the origin and with "semi-latus rectum" length is a. From this focal point, a ray is extended to intersect the curve at a variable point P, which gradually moves along the parabola toward the vertex while completing a full angular sweep. This movement exhibits periodic behavior, generating cyclic variations in the ray's length, which is the template for the generalized "Pythagorean relation" of these new functions. Building on parabola definitions, the distance between the parabola's focus and the directrix (a), which is here a constant, used in all equations to build all mathematical justifications (Picture 1).



Picture 1. Plot of the geometric method used in this paper to produce the parabolic functions, where $P(x_p,\ y_p)$ at the parabola segment defined by angle u ranged within; ($u \leq \pi$)

Furthermore, the methodology includes calculating the values of the primary trigonometric functions and their reciprocals at each angle, standardizing the length of the chord (the ray from the focus to the curve) for consistency. All equations presented in this paper, from Equation 1 to Equation 78, are derived from the geometric method illustrated in Figures 1, 2, and 3, as well as these in Appendix A, (Picture A1–A5). These visual representations are used to describe the geometric proportions underlying the parabolic functions. Specifically, they illustrate how the coordinates of a point $P(x_p, y_p)$ on the parabola relate to the focus and the reference angle u, forming the basis for defining the parabolic sine and cosine functions. As a result, the derived identities and expressions consistently reflect the spatial relationships encoded in the geometry of the parabola, ensuring that each function is grounded in both analytic and visual reasoning. This method allows for precise determination of the coordinates of the intersection point P at any given angle of inclination u of the ray PF. Additionally, the new arc referred to as the *Sinocon Arch* is constructed with its center at the parabola's focus,F, and a radius equal to twice the fixed distance between the focus and the parabola's axis of symmetry (the Directrix), the *radius* = 2a. The *Sinocon* Arch serves as a reference for analyzing the parabolic curve's dimensions as the point of intersection varies within a defined angular range ($u \leq \pi$), as it is illustrated in Picture 1.

The target here is to obtain results that clearly reflect the specific relationships and geometric parameters within the framework of the developed parabolic trigonometric functions. In particular, the coefficients of the segment from any point (x_p, y_p) on the parabola to the focus, $\overline{FP}$ with angle uuu in the integrals, reflect different ratios related to x_p and y_p. Consequently, these coefficients indicate how the rate of change of the integral with respect to uuu is influenced by the parabola's parameters.

Moreover, the logarithmic terms used in the integrals contain different coefficients corresponding to x_p and y_p, respectively. This implies that the contribution of the logarithmic term to the overall value of the integral is scaled differently for *sinp u* and *cosp u*. Although the argument inside the logarithm remains the same in both integrals, the coefficients outside the logarithm vary, highlighting the distinct roles of each parameter. Additionally, both integrals include the constant a, which accounts for the indefinite nature of integration. Finally,the coefficients are determined by y_p, for *sinp u*, and by x_p, for *cosp u*, thereby reflecting their respective influences.

Results and Discussions

This study utilized the geometric development model of parabola functions, which consists of six rtigonomertry functions, where P(x_p, y_p) on the parabola segment defined by angle u which is ranged within; ($u \leq \pi$). For any angle u, the ray $\overline{FP}$ will intersect the parabola segment at P with projection of x_p and y_p, while when this ray intersects the Arch, it determines the parabolic functions, see Picture 2.

Definition of Parabolic Identities

The general idea in this paper is obtaining explicit formulas for a point $P(x_p, y_p)$ which is located on the parabola segment consists in Picture 3. Consider a parabola segment with focus F at the origin point (0, 0), where a is the distance between F and the vertex of the parabola, hence the standard form has the vertex on the x-axis at the point (-a, 0), and the parabola directrix the line with equation; $x = -a$, lies at x-axis from the vertex of the parabola. Note that value of angle (u) at focus is a parameter of point $P(x_p, y_p)$ and then the length of line segment $\overline{FP}$.

Let F is the origin, (0,0), and P(x_p, y_p) is a point at the parabola segment where the distance from the focus and the vertx is a. Then construct a ray from the parabola focus and point P, $\overline{FP}$ with angle u, then the perpendicular from P at the x-axis is $\overline{FP}$, at point F draw $\overline{FP}$.

For a parabola segment with the standard form, $y^2 = 4ax$ (where a is a constant that determines the distance between the vertex and the focus), from the right angled triangle ΔFPD, the distance $\overline{FP}$ between the focus F and a point $P(x_p, y_p)$ on the parabola is given by:

$$\overline{FP} = \sqrt{(x_p - a)^2 + y_p{}^2}\,, \tag{1}$$

Then substitute $y_p = \sqrt{4ax_p}$, into the equation to find PF purely in terms of *x* and a. Consequently, the ray length $\overline{FP}$ can be expressed as:

$$\overline{FP} = \sqrt{(x_p - a)^2 + 4ax_p}\,, \tag{2}$$

Expanding and simplifying the expression under the square root:

$$\overline{FP} = \sqrt{x_p{}^2 - 2ax_p + a^2 + 4ax_p}\,, \tag{3}$$

$$\overline{FP} = \sqrt{x_p{}^2 + 2ax_p + a^2}\,, \tag{4}$$

This can also be factored to:

$$\overline{FP} = \sqrt{(x_p + a)^2}\,, \tag{5}$$

In a parabola, the distance from the point to the focus equals the distance from the point to the directrix. Since $\overline{FP}$ represents a distance, which is always non-negative:

$$\overline{FP} = |x_p + a|\,, \tag{6}$$

From the point P(x_p, y_p), draw a perpendicular line that intersects the x-axis at point D. The triangle $\triangle y_p x_p$FP is right-angled, with x_p on the x-axis, $\overline{FP}$ as the hypotenuse, and y_p as the vertical side. From this, we will derive the trigonometric functions which represents the ratio of the vertical side y_p and horizontal side x_p to the hypotenuse $\overline{FP}$ related the point S along the parabolic segment, thus:

- Parabola sine function (*sinp u*): This represents the ratio of the vertical side x_p to the hypotenuse $\overline{FP}$.
- Cosine Function (*cosp u*): This represents the ratio of the base y_p, to the hypotenuse $\overline{FP}$.

These functions describe the relationship between the coordinates of the point P(x_p,y_p), on the parabola and the angle u formed at the focus F of the parabola. From the right-angled triangle (ΔFPC), we find the two fundamental functions of the parabola; $sinp\, u$ and $cosp\, u$, which can be derived for any point, P(x_p, y_p) is a point at the parabola segment:

$$sinp\, u = \frac{y_p}{|x_p \pm a|}\,, \tag{7}$$

$$cosp\, u = \frac{x_p}{|x_p \pm a|}\,, \tag{8}$$

Also, Arch Center point is the parabola focus, which lies at the origin point, which is the point F (0,0), then, the ($tanp\ u$) can be defined as a ratio by the tangent ($\overline{HA}$) of the unit circle. Hence the tangent of an angle u is defined as the ratio of the $sinp\, u$ to the $cosp\ u$ of the angle, and then the proportion:

$$tanp\ u = \left(\frac{y_p}{x_p}\right), \tag{9}$$

And from the definition of the tangent function, we can derive:

$$tanp\ u = \left(\frac{sinp\ u}{cosp\ u}\right), \tag{10}$$

Thus:

$$cotp\ u = \left(\frac{x_p}{y_p}\right), \tag{11}$$

To find the reciprocal of the parabolic sine function (7):

$$\frac{1}{sinp\, u} = \frac{|x_p \pm a|}{y_p},$$

Therefore, the reciprocal of the parabolic sine function $sinp\, u$ is $cscp\ u$:

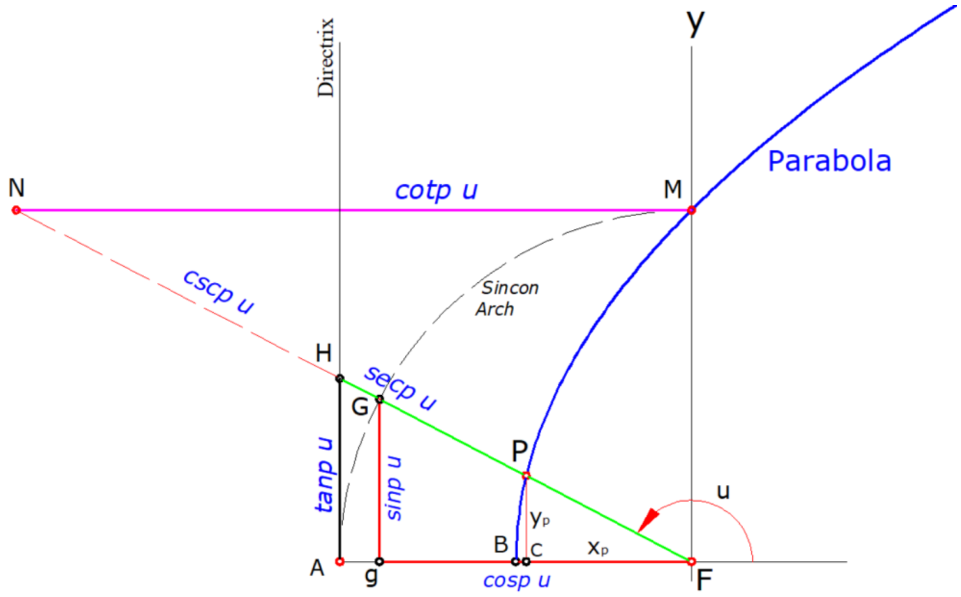
$$cscp\ u = \frac{|x_p \pm a|}{y_p}, \tag{12}$$

Thus the reciprocal of the parabolic cosine function $cosp\ u$ is $secp\ u$:

$$secp\ u = \frac{|x_p \pm a|}{x_p}, \tag{13}$$

It is important to note that a standard parabola, $y^2 = 4ax$ has a distinct relationship between x_p and x_p, while $|x_p \pm a|$ doesn't directly describe a parabolic curve but rather a ratio that depends on the coordinates relative to the focus. These expressions provide the reciprocal trigonometric functions based on the parabolic coordinates. These functions are expressed in a normalized form relative to the horizontal distance $|x_p \pm a|$. This normalization provides a clear and specific way to relate the coordinates of a point on the parabola to trigonometric functions. Understanding how the coordinates of a point on the parabola relate to trigonometric measures.

Also, applying these relationships in contexts where parabolic shapes are used, such as in optics or structural design. Graphing these functions can visualize how each function behaves with respect to the coordinates on the parabola, Picture 2.



Picture 2. Plot of the geometric method to draw the parabolic functions when angle u is greater than $\frac{\pi}{2}$

When $\left(\frac{\pi}{2} > u \leq \pi\right)$*, the parabolic functions are:*

In this case, when angle u greater than $\left(\frac{\pi}{2}\right)$, the ray $\overline{FP}$ extends to intersect the Directrix at point H, thus:

$$\overline{FP} = \overline{FH} = secp\, u\,, \tag{14}$$

It is seen from the function proportions in Picture 3 that because the ray $\overline{FH}$ lies within the negative part of x-axis, then the value of $cosp\ u$ and then x_p are negative, (see Table 1). Also, at this range of u, it can lead to obtain another relationship from the right-angled triangle ΔHAF, that connect between parabolic functions of $secp\ u$ and $tanp\ u$:

$$secp\ u = \sqrt{tanp^2 u + 4a^2}\ , \tag{15}$$

And by rewriting the given equation in terms of $secp\ u$ and $sinp\ u$:

$$secp^2\ u = tanp^2 u + 4a^2\ , \tag{16}$$

Hence $cscp\ u$ is the reciprocal of $sinp\ u$. To express $cscp\ u$ using the given relationship, we use the identity:

$$cscp\ u = \frac{secp\ u}{tanp\ u}, \tag{17}$$

Thus, use these identities to express $sinp\ u$ and $cosp\ u$ in terms of $tanp\ u$ and a:

$$cscp\ u = \frac{\sqrt{tanp^2 u + 4a^2}}{tanp\ u}, \tag{18}$$

In essence, you've used the given equation to derive expressions for *cscp u* and *cotp u* based on their relationships with *tanp u* and *secp u*. These expressions help us understand how these modified trigonometric functions relate to each other in the context of specific geometric problem involving the parabola.

Algebraic Values:

For a parabola with (a – 1), the algebraic expressions for the most important angles are as follows:

The equation simplifies to $(y_p)^2 = 4\, x_p$, so that; $x_p = \frac{(y_p)^2}{4}$, then this form required to calculate x_p and y_p. based on these trigonometric identities. For each angle u, computed by the following forms:

$$|x_p \pm a| = \sqrt{(x_p)^2 + (y_p)^2}\ , \tag{19}$$

Thus:

$$sinp\ u = \frac{y_p}{\sqrt{(x_p)^2 + (y_p)^2}}\ , \tag{20}$$

$$cosp\ u = \frac{x_p}{\sqrt{(x_p)^2 + (y_p)^2}}\ , \tag{21}$$

Also,

$$cscp\ u = \frac{\sqrt{(x_p)^2 + (y_p)^2}}{y_p}\ , \tag{22}$$

Then:

$$secp\ u = \frac{\sqrt{(x_p)^2 + (y_p)^2}}{x_p}\ , \tag{23}$$

Building on the previous results, and from any point P(x_p, y_p), on the parabola segment , the values of x_p, and y_p can be determined according to values of distance of $\overline{BA}$ and values of *sinp u* and *cosp u*, by formulae :

$$y_p = |x_p \pm a|.sinp\,u\ , \tag{24}$$

$$x_p = |x_p \pm a|.cosp\,u\ , \tag{25}$$

Also, at this range of slops, $\left(\frac{\pi}{2} > u \leq \pi\right)$, this leads to obtain other traigonometrci relationships from the right-angled triangle Δ HAF, that:

$$2a = \sqrt{sinp^2u + cosp^2u}\ , \tag{26}$$

$$sinp\,u = \sqrt{4a^2 + cosp^2u}\ , \tag{27}$$

And according to the trigonometric ratios, we find the following:

$$\frac{sinp\,u}{y_p} = \frac{2a}{\overline{GP}}\ , \tag{28}$$

$$\overline{GP} = \frac{y_p.2a}{sinp\,u}\ , \tag{29}$$

Which can be presented by:

$$\overline{GP} = \frac{2a\,.\,|x_p \pm a|.sinp\,u}{sinp\,u}\ , \tag{30}$$

Thus, length value of line segment is given by:

$$\overline{GP} = 2a.\,|x_p \pm a|\ , \tag{31}$$

And from the right-angled triangle Δ HAF, (Picture 3), the length of $\overline{FP}$ is:

$$\overline{FP} = 2a - \left(\frac{y_p\,.2a}{sinp\ u}\right)\ , \tag{32}$$

$$\overline{FP} = 2a - 2a\,.\,|x_p \pm a|\ , \tag{33}$$

and then:

$$\overline{FP} = 2a\left(1 - |x_p \pm a|\right), \tag{34}$$

Also, where P(x_p,y_p) is a point at the parabolic segment, and a is the distance between the parabola focus and the vertex, thus the value of a is remain as a constant.

The Sincon's Arch

In this paper, a special arch is designed at a quarter circle whose center is the parabola focus point, F, lies at the origin point, which is the point F (0, 0). The *Sinocon's Arch* is constructed with its center at the parabola's focus, F, and a radius equal to twice the fixed distance between the focus and the parabola's axis of symmetry (*the Directrix*), the radius = $2a$. The *Sinocon's* Arch serves as a reference for analyzing the parabolic curve's dimensions as the point of intersection varies within a defined angular range (u ≤ π), as it is illustrated in Picture 1. In this paper, the given name of *Sincon's Arch* is a mix of sine and cosine words, and it is drawn from the focus point with radius of $2a$, the basic geometric specifications for *Sincon's Arch* are outlined in the following Table 1.

Table 1. Key proportions of *Sincon's* Arch

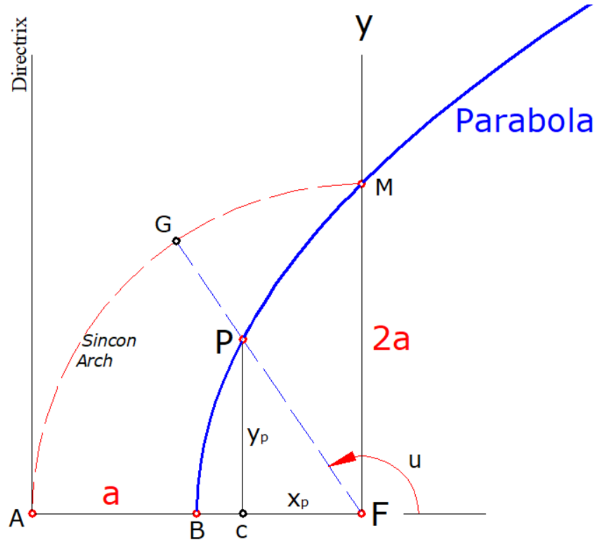
Proportions	Values
Start Point at y-axis	$(0,\pm 2a)$
End Point at x-axis	$(-2a,0)$
Arch Center point, the Origin point.	F= (0,0)
Arch's radius	$2a$
Arch length	$1.5708a$
Shaded area by the arch.	$0.785375a$
Where (a) is a constant, which is the distance from the parabola's focus and the vertex.	

There is a fundamental aspect of designing *Sincon's* Arch with these specifications, which are linked with values of *sinp u* and *cosp u*. The purpose of designing this arc is evident in its use as a reference for determining the extension points of both *sinp u* and *cosp u*, which result from the intersection point of the ray with the arc's circumference. From this intersection point, G, a perpendicular line is drawn to the negative x-axis. The length of this perpendicular line $\overline{Gg}$ determines the value of the $sinp\, u$, while the distance between the projection of this line and the horizontal extension to the focus point, $\overline{Fg}$, gives the value of the $cosp\ u$, where:

$$\overline{AB} = \overline{BF} = \frac{1}{\overline{MF}} = a\ , \tag{35}$$

$$\overline{FG} = \overline{MF} = \overline{FA} = 2a\ , \tag{36}$$

There is a constant ratio between the area under the parabola curve and the remaining area occupied by the arc (AGM), which remains consistent regardless of the curve's condition, as it is illustrated in Picture 4.



Picture 4. Plot the proportions of the *Sincon's* Arch related to the parabola segment and Unit crcle at origin point, F, where P(x_p, y_p), is a point on the parabola segment

Construction Method

In this paper, a geometric method is used based on proprertions of parabola and the unit circle, to represent these driven six functions associated with the parabolic segment, (*sinp u*, *cosp u*, *tanp u*, *secp u*, *cscp u*, and *cotp u*), based on the following key technique:

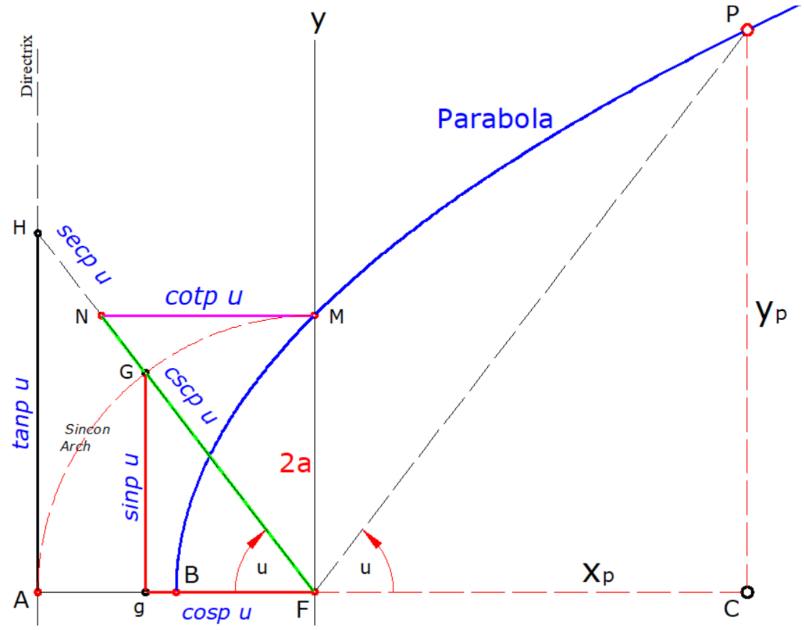
By drawing a parabola segment such that its focus, F, is positioned at the origin point, F(0, 0), see Picture 4 .

- From the parabola diffenation, draw the Directrix from point A, parallel to the y-axis and at a horizontal distance of a from the focus, F.
- Identify a point on the parabola curve, $P(x_p, y_p)$, and then construct a ray connecting this point to the focus, $\overline{FP}$.
- The vertical projection of the point onto the x-axis, represents the value of y_p, while the horizontal distance from the focus, represents the value of x_p.
- Draw an arch (named as *Sincon*-Arch), whose center is the focus,F, with radius of $2a$, this arch intersects the y-axis at the same point, G, in which the parabola crosses the y-axis.

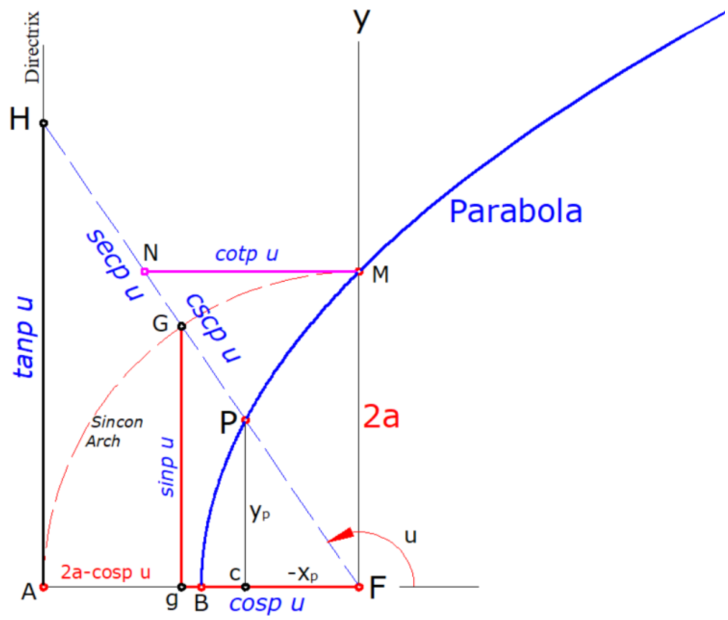
After determining the coordinates, reflect the ray $\overline{FP}$ image at the same angle of inclination but on the left side of the y-axis, i.e., in the second quadrant. This is a crucial step to enable the determination of the values of the six trigonometric functions for the parabola. The intersection point of the reflected ray on the Directrix, H, will determine the value of *tanp u*, and the intersection point of the reflected ray on the *Sincon*-Arch, G, will determine the value of *sinp u*. The horizontal distance from the focus to the position of the vertical projection of the intersection point of the reflected ray with the *Sincon*-Arch, will represent the *cosp u*, where:

- When $\left(0 > u \leq \pm\frac{\pi}{2}\right)$, the $secp\ u = \overline{HF}$, and $cscp\ u = \overline{HF}$, where$(secp\ u\ > cscp\ u\ )$.
- When $\left(\frac{\pi}{2} > u \leq \pm\ \pi\right)$, the $secp\ u$, and $cscp\ u = \overline{HF}$, where $(secp\ u < cscp\ u\ )$.

According to the innovative geometric method introduced in this paper, after drawing the ray at a specific angle, the direction of the (FP), ray is reflected as a mirror image with the same angle of inclination at the negative *x*-coordinate, extending to intersect the parabola's Directrix at the extension point H.



When: $\left(0 > u \leq \frac{\pi}{2}\right)$,



When: $\left(\frac{\pi}{2} > u \leq \pm\, \pi\right)$,

Picture 4. Plot of the geometric method to draw the parabolic functions related to angle u values, where P(x_p, y_p) is a point on the parabola segment (PF), and the distance from the focus (F) and the vertex (AB = BF) is *a constant =a*, while the ray $\overline{FP}$ slops with angle u

The horizontal distance between the intersection point, (G), of the parabolic curve with the y-axis, extended to its intersection with the ray $\overline{FH}$, represents the reciprocal of the tanp u value. The key steps for determining the trigonometric ratios specific to the six functions of the parabola are listed in Table 2, and as shown in Picture 4.

Table 2. Plot the parabolic functions derived by a point P (x_p, y_p) on the parabola.

Parabolic	***sinp u***	***cosp u***	***tanp u***	***secp u***	***cscp u***	***cotp u***
Line Segments	Gg	Fg	HA	HF	NF	NM

Where, a is the distance between the focus and the vertex, and u is the angle of the ray FP

Visualizing the Parabolic Functions

It can be observed from figure 8 that for every $\overline{PF}$ with an angle less than $\left(\frac{\pi}{2}\right)$ degrees, there is a corresponding negative angle that shares the same parabolic functions of sine, cosine, tangent, and their reciprocals. The only difference is by values of x_p and y_p because of position of point P related the parabola segment. For example, when $\overline{PF}$ lies by u = 56º, hence this angle is a corresponding the angle: $(2\pi - u) =$ 124º, as they were listed in Tables 3 and 4.

Table 3. The parabolic functions with $a = 1$, and for angles u ranged from $(u \leq \pi)$

Angle *u*	x_p	y_p	***sinp u***	***cosp u***	***tanp u***	***secp u***	***cscp u***	***cotp u***
0	∞	0.000000	0.000000	a	0.000000	1	∞	0.000000
5	305.8683	24.8189	0.0809	0.9967	0.0808	1.00331	12.3609	12.37623
10	81.2625	13.7386	0.1667	0.9860	0.1691	1.01419	5.99880	5.913660
15	28.1615	7.73930	0.2650	0.9643	0.2748	1.03702	3.77358	3.639010
20	15.6294	5.6740	0.3413	0.9400	0.3630	1.06382	2.92997	2.754820
25	9.9238	4.5712	0.4184	0.9083	0.4606	1.10095	2.39000	2.171081
30	6.3325	3.6974	0.5042	0.8636	0.5838	1.15794	1.98333	1.712915
35	4.5893	3.1883	0.5706	0.8213	0.6947	1.21758	1.75254	1.439470
40	3.2592	2.7329	0.6425	0.7663	0.8385	1.30497	1.55642	1.192605
45	2.4412	2.4140	0.7031	0.7111	0.9888	1.40627	1.42227	1.011326
50	1.7628	2.1203	0.7690	0.6393	1.2028	1.56421	1.30039	0.831393
55	1.3130	1.9011	0.8228	0.5683	1.4479	1.75963	1.21536	0.690655
60	1.0159	1.7397	0.8636	0.5043	1.7124	1.98294	1.15794	0.583975
65	0.7087	1.5543	0.9099	0.4148	2.1933	2.41080	1.09902	0.455933
70	0.5153	1.4250	0.9404	0.3401	2.7654	2.94031	1.06337	0.361611
75	0.3433	1.2988	0.9668	0.2556	3.7828	3.91236	1.03434	0.264354
80	0.2164	1.1972	0.9841	0.1779	5.5327	5.62113	1.01615	0.180743
85	0.0987	1.0944	0.9960	0.0898	11.0882	11.1358	1.00401	0.090185
89	0.0224	1.0221	0.9998	0.0219	45.6885	45.66210	1.00020	0.021887
90	0.000000	$2a$	$2a$	0.000000	∞	∞	a	∞
95	-0.0759	0.9223	0.9966a	-0.0820	12.1514	12.1951	1.00341	0.082295
100	-0.1481	0.8394	0.9848	-0.1738	5.66780	5.75373	1.01543	0.176435
105	-0.2042	0.7691	0.9665	-0.2566	3.76620	3.89711	1.03466	0.265519
110	-0.2558	0.6987	0.9391	-0.3438	2.73170	2.90866	1.06485	0.366072
115	-0.2951	0.6401	0.9081	-0.4187	2.16880	2.38834	1.10120	0.461084
120	-0.3354	0.5739	0.8634	-0.5046	1.71110	1.98176	1.15821	0.584419
125	-0.3617	0.5259	0.8240	-0.5667	1.45400	1.76460	1.21359	0.687758

Angle *u*	x_p	y_p	*sinp u*	*cosp u*	*tanp u*	*secp u*	*cscp u*	*cotp u*
130	-0.3894	0.4701	0.7701	-0.6379	1.20720	1.56764	1.29853	0.828363
135	-0.4122	0.4187	0.7126	-0.7016	1.01560	1.42531	1.40331	0.984639
140	-0.4332	0.3659	0.6453	-0.7640	0.84470	1.30890	1.54966	1.183852
145	-0.4491	0.3189	0.5789	-0.8154	0.71000	1.22639	1.72741	1.408450
150	-0.4648	0.2665	0.4974	-0.8675	0.57340	1.15273	2.01045	1.743983
155	-0.4754	0.2213	0.4220	-0.9066	0.46540	1.10302	2.36967	2.148689
160	-0.4838	0.1736	0.3377	-0.9412	0.35880	1.06247	2.96120	2.787068
165	-0.4914	0.1308	0.2573	-0.9663	0.26610	1.03487	3.88651	3.757985
170	-0.4968	0.0800	0.1591	-0.9873	0.16110	1.01286	6.28535	6.207324
175	-0.4990	0.0449	0.0896	-0.9960	0.08990	1.00401	11.16071	11.123470
180	-*a*	0.000000	0.000000	-2*a*	0.000000	-*a*	∞	∞

Values for x_p and y_p are rounded to 4 decimal places, and 6 digits for *secp u*, *cscp u*, and *cotp u*.
Parabolic functions are calculated based on the angle u and corresponding x_p and y_p.
For angles approaching $\frac{\pi}{2}$ degrees, the tangent and secant functions approach infinity.

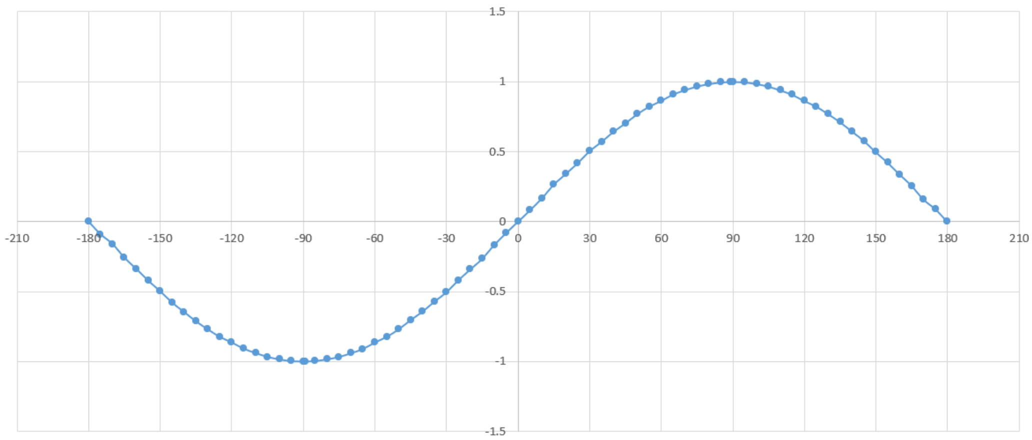
Table 4. The parabolic functions with *a* = 1, and for angle *u* ranged from (u ≤ π)

Angle *u*	x_p	y_p	*sinp u*	*cosp u*	*tanp u*	*secp u*	*cscp u*	*cotp u*
5	305.8683	24.8189	0.0809	0.9967	0.0811	1.0033109	12.360939	12.330456
175	-0.49900	0.0449	0.0809	0.9967	0.0811	1.0033109	12.360939	12.330456
10	81.2625	13.7386	0.1667	0.9860	0.1691	1.0141987	5.998800	5.9136605
170	-0.4964	0.0839	0.1667	0.9860	0.1691	1.0141987	5.998800	5.9136605
15	28.1615	7.7393	0.2650	0.9643	0.2748	1.0370216	3.7735849	3.6390101
165	-0.4909	0.1349	0.2650	0.9643	0.2748	1.0370216	3.7735849	3.6390101
20	15.6294	5.6740	0.3413	0.9400	0.3630	1.0638297	2.9299736	2.7548209
160	- 0.4845	0.1759	0.3413	0.9400	0.3630	1.0638297	2.9299736	2.7548209
25	9.9238	4.5712	0.4184	0.9083	0.4606	1.1009578	2.3900573	2.1710812
155	-0.4760	0.2192	0.4184	0.9083	0.4606	1.1009578	2.3900573	2.1710812
30	6.3325	3.6974	0.5042	0.8636	0.5838	1.1579434	1.9833399	1.7129153
150	-0.4634	0.2706	0.5042	0.8636	0.5838	1.1579434	1.9833399	1.7129153
35	4.5893	3.1883	0.5706	0.8213	0.6947	1.2175818	1.75254118	1.4394702
145	-0.4509	0.3133	0.5706	0.8213	0.6947	1.2175818	1.75254118	1.4394702
40	3.2592	2.7329	0.6425	0.7663	0.8385	1.3049719	1.55642023	1.1926058
140	-0.4338	0.3638	0.6425	0.7663	0.8385	1.3049719	1.55642023	1.1926058
45	2.4412	2.4140	0.7031	0.7111	0.9888	1.4062719	1.42227279	1.0113268
135	-0.4155	0.4109	0.7031	0.7111	0.9888	1.4062719	1.42227279	1.0113268
56	1.2847	1.8890	0.8269	0.5624	1.4705	1.7780938	1.20933607	0.6800401
124	-0.3599	0.5293	0.8269	0.5624	1.4705	1.7780938	1.20933607	0.6800401
60	1.0159	1.7397	0.8636	0.5043	1.7124	1.9829466	1.15794349	0.5839757
120	-0.3352	0.5741	0.8636	0.5043	1.7124	1.9829466	1.15794349	0.5839757
65	0.7087	1.5543	0.9099	0.4148	2.1933	2.4108003	1.09902187	0.4559339
115	-0.2932	0.6431	0.9099	0.4148	2.1933	2.4108003	1.09902187	0.4559339
70	0.5153	1.4250	0.9404	0.3401	2.7654	2.9403116	1.06337728	0.3616113
110	-0.2538	0.7018	0.9404	0.3401	2.7654	2.9403116	1.06337728	0.3616113
80	0.2164	1.1972	0.9841	0.1779	5.5327	5.6211354	1.01615689	0.1807435
100	-0.1510	0.8355	0.9841	0.1779	5.5327	5.6211354	1.01615689	0.1807435
90	0.0000	2*a*	2*a*	0.0000	∞	∞	*a*	∞
180	-*a*	0.0000	0.00000	-2*a*	0.00000	-*a*	∞	∞

To visualize the Parabolic Functions, we will graph the sine and cosine functions for a point P(x_p, y_p), moving along the parabola segment. The angle u will change as x_p changes. Similarly, the parabolic sine function $\frac{y_p}{|x_p \pm a|}$, will typically increase as y_p increases, and will be inversely proportional to $|x_p \pm a|$. While Parabolic cosine function $\frac{x_p}{|x_p \pm a|}$, will typically increase as x_p increases, and will be inversely proportional to $|x_p \pm a|$. To graph these parabolic functions, we need to:

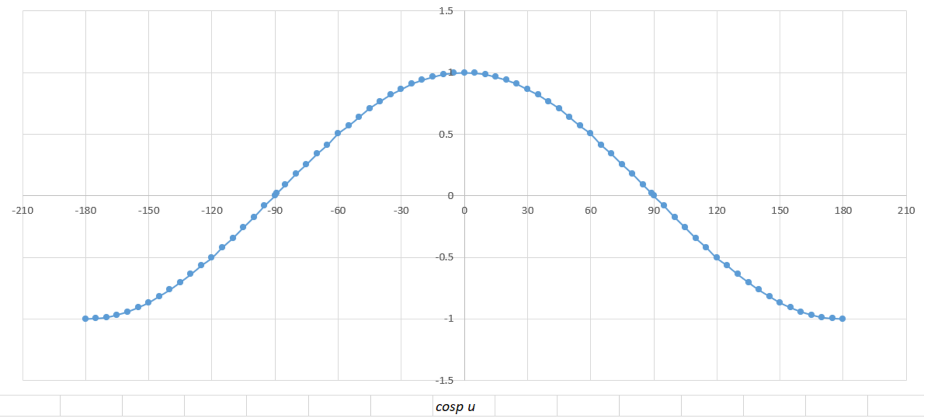
1. Calculate functions for various angles.
2. Plot parabolic functions versus angle u.

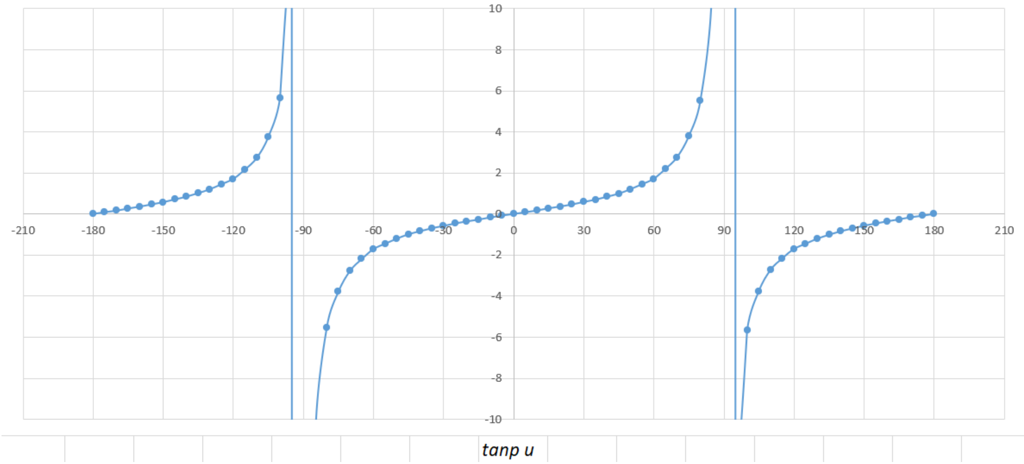
And by the *Desmos*, here is the graph of the parabolic sine function $sinp\, u$ versus the variable x_p, with a fixed value of a = 1. The function is defined as: $sinp\, u = \frac{y_p}{|x_p - a|}$. The graph below shows how the parabolic sine function behaves with respect to changes in x_p. The plot demonstrates the relationship between the input x_p and the corresponding value of s $sinp\, u$, see Pictures 5 to 8.



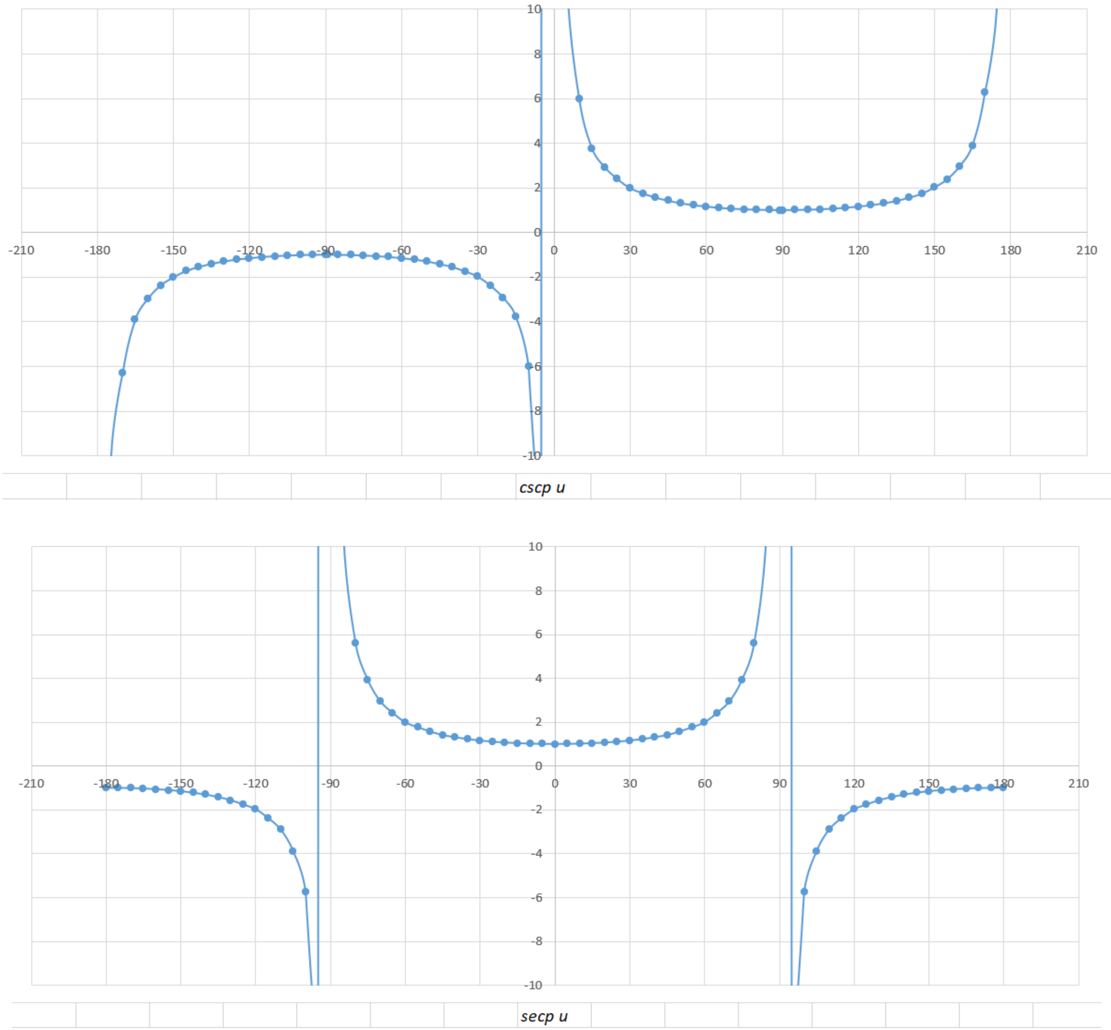
Picture 5. Graph of the parabolic sine function versus angle by the *Desmos*.

Similerlly, the graphs of all parabolic functions are plotted below:

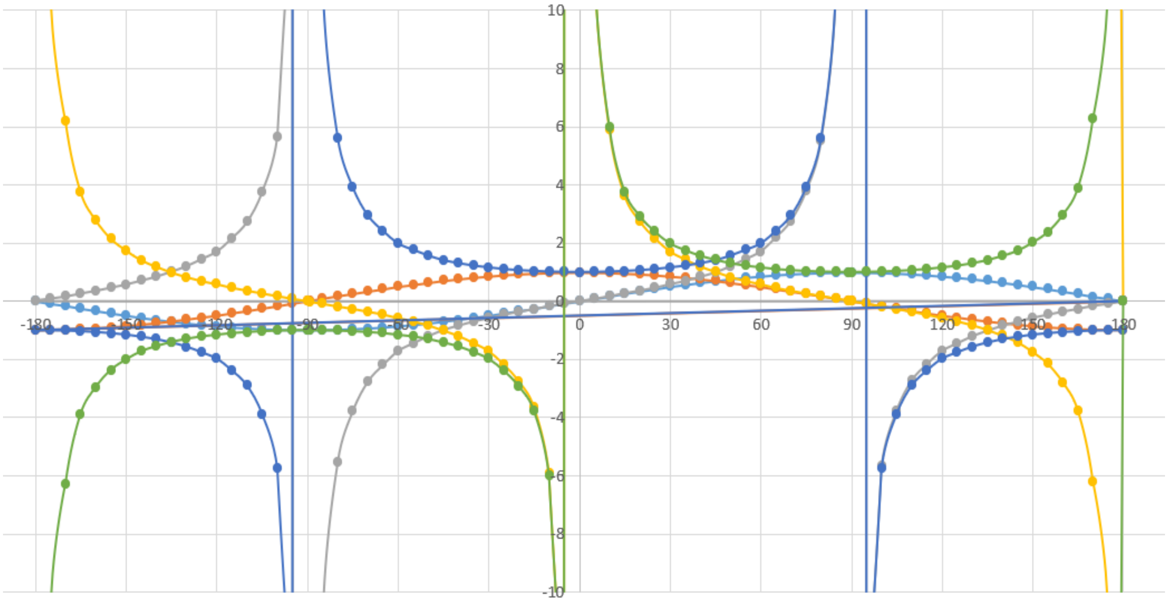




Picture 6. Graphs visualize by the *Desmos* the behaviour of *cosp u*, *tanp u*, and *cotp u* functions with respect to the variables, u, x_p, and y_p, where $a = 1$. The plots also illustrate how these parabolic functions relate to each other



Picture 7. By the *Desmos*, visualizing the behaviour of *csp u*, and *secp u*, parabolic functions with respect to the variable, u, x_p and y_p, where $a = 1$



Picture 8. By the *Desmos*, visualizing the behaviour of all 6 parabolic functions, to the variables, u, x_p and y_p, where a = 1

Integration of Parabolic Functions

Let's integrate each of these functions with respect to u. Note that these integrals might not always have straightforward antiderivatives, especially if the functions are defined in terms of x_p and y_p, which are functions of u. Here, we'll perform the integrations assuming the a is constants and parameters x_p and y_p are functions of u.

$$\int sinp\, u\, du = \int \frac{y_p}{|x_p \pm a|}\, du = \frac{y_p}{|x_p \pm a|} + C, \tag{37}$$

Perform Integration using standard techniques or numerical methods if necessary. Since a is the constant distance between the parabola's focus and its directrix, it plays a significant role in the equations related to the parabolic functions. The integral must represent the parabolic function's relation to the geometric configuration of the parabola. The constant the constant a shift the function, reflecting the geometry of the parabola. To integrate the parabolic *sine*-function, *sinp u*:

Given $x_p = |x_p \pm a|\,.cosp\, u$, and $y_p = |x_p \pm a|.sinp\, u$, these can be written as:

$$\int sinp\, u\; du = \int \frac{y_p}{|x_p \pm a|}\, du = \int \frac{y_p}{\left||x_p \pm a|\,.cosp\, u \pm a\right|}\, du, \tag{38}$$

$$\int cosp\, u\; du = \int \frac{x_p}{|x_p \pm a|}\, du = \int \frac{x_p}{\left||x_p \pm a|\,.sinp\, u\, \pm a\right|}\, du, \tag{39}$$

Note that these integrals represent the relationship between the angle u and the parabolic sine and cosine function, taking into account the geometric properties of the parabola. To simplify this advanced integral, we need to use some advanced integration techniques, such as variable substitution or series expansion to analyze complex functions like $sinp\ u$. We start by attempting to analyze the complex function within the absolute value. Let's assume:

$$z = |x_p \pm a|.cosp\ u \ \pm a\ ,$$

So, the integral becomes:

$$\int \frac{y_p}{|z|}\ du\ , \tag{40}$$

Assume that z can be simplified or changed to a new variable v where:

$$v = |z|\ , \tag{41}$$

This implies:

$$dv = \frac{d}{du}\left(|x_p \pm a|\ .cosp\ u \ \pm a\right)du\ , \tag{42}$$

If we assume that $cosp\ u$ can be represented similarly to traditional trigonometric functions, we can use derivatives of $cosp\ u$ to calculate dv. If the function $cosp\ u$ has an expression that can be expanded as a series, we can use a Taylor series (Chemin, 2005; Fahim et al., 2021), to approximate the function. For example:

$$cosp\ u = \sum_{n=0}^{\infty} c_n\ u^u\ , \tag{43}$$

Where c_n are coefficients depending on the properties of the function. We can then substitute this expression into the integral and analyze it term by term, then evaluate the integral, we get:

$$\int \left(\frac{y_p}{v}\right) .\frac{dv}{du}\ du = y_p \int \left(\frac{1}{v}\right) dv\ , \tag{44}$$

This gives:

$$y_p \ln|v| + C_2\ , \tag{45}$$

Substitute the Original Variables:

We substitute *v* back with the original value, we assumed for *z*:

$$y_p \ln\left||x_p \pm a|.cosp\, u \ \pm a\right| \ + C_2 \,, \tag{46}$$

Thus, the integral simplifies to:

$$\int sinp\, u \ du = \frac{y_p}{|x_p \pm a|} \ .u + y_p \ln\left||x_p \pm a|.cosp\, u \ \pm a\right| \ + C \,, \tag{47}$$

Where C is the final constant of integration, and thus, the integral of $cosp\ u$ is:

$$\int cosp\, u \ du = \frac{x_p}{|x_p \pm a|} \ .u + x_p \ln\left||x_p \pm a|.cosp\, u \ \pm a\right| \ + C \,, \tag{48}$$

To find the integral of *cscp u* (*cosecant*) based on the integral of $sinp\, u$ and its relationship with *cosp u*, we typically use the integral formula:

$$\int cscp\, u\, du = \ - \ \ln \ | \ cscp\, u + cotp\, u \,| + C \,, \tag{49}$$

The integral of $cscp\ u$ can be written as:

$$\int cscp\, u\, du = \ - \ \ln \left| \frac{sinp\, u + cosp\, u}{sinp\, u} \right| + C \,, \tag{50}$$

Given the specific form of the integral for $sinp\, u$:

$$\int sinp\, u \ du = \frac{y_p}{|x_p \pm a|} \ .u + y_p \ ln\left||x_p \pm a|.cosp\, u \ \pm a\right| \ + C \,, \tag{51}$$

To find *cscp u*, substitute and rearrange:

$$\int cscp\, u\, du = \ - \ \ln \left| \frac{sinp\, u + \frac{sinp\, u + cosp\, u}{sinp\, u}}{sinp\, u} \right| + C \,, \tag{52}$$

Simplify to:

$$\int cscp\, u\, du = \ - \ \ln \ | \ cscp\, u + cotp\, u \,| + C \,, \tag{53}$$

Incorporating the specific parameters, the integral might take a form that aligns with your definitions of x_p and y_p, and other constants. In the context of *cscp u*, which is the cosecant function related to the parabola, the same general formula applies. Similarly, the integral of *secp u* would be:

$$\int secp\, u\, du = \ln |\, secp\, u + tanp\, u\,| + C\,, \tag{54}$$

This result reflects the integral's form, adjusted for the parabolic secant function, *secp u*. This simplification heavily depends on the properties of the function *cosp u* and whether it can be represented by series or known derivatives. These results reflect the specific relationships and geometric parameters in the context of parabolic trigonometric functions. The coefficients of angle *u* in the integrals reflect different factors related to x_p and y_p. These coefficients indicate how the rate of change of the integral with respect to *u* is affected by the parameters of the parabola. The logarithmic terms have different coefficients, to x_p and y_p, respectively. This suggests that the contribution of the logarithmic term to the integral's overall value is scaled differently for *sinp u* and *cosp u*. The argument inside the logarithm is the same for both integrals, but the coefficients outside the logarithm vary. Both integrals include the constant C, which accounts for the indefinite nature of the integration. The coefficients are determined by yp for *sinp u* and x_p for *cosp u*. This reflects the different influence of these parameters in the context of the integrals. Also, both integrals have a logarithmic term involving $|x_p \pm a|$ *cosp u* $\pm a$, but with different scaling factors.

The idea of an infinite product expansion for parabolic functions *sinp u* and *cosp u* based on their definitions could be analogous to the infinite product expansions of standard trigonometric functions. The infinite product expansions for standard trigonometric functions are based on their zeroes or roots. For parabolic trigonometric functions, which are defined analogously but relate to a parabola, a similar approach can be adapted.

Given:

$$sinp\, u = \frac{y_p}{|x_p \pm a|},$$

$$cosp\, u = \frac{x_p}{|x_p \pm a|},$$

Now, to explore potential infinite product expansions for these functions. Infinite Product Expansion for *sinp u*: For standard sine, the infinite product is:

A similar approach could be imagined for *sinp u*, involving the parameters y_p and $|x_p \pm a|$. We would express *sinp u* in a form that captures its dependence on *u*.

$$sinp\, u = u \prod_{n=1}^{\infty} \left(1 - \frac{u^2}{n^2 \pi^2}\right), \tag{55}$$

Infinite Product Expansion for *cosp u*: For the standard cosine, the infinite product is:

An analogous expansion for *cosp u* would similarly involve x_p and $|x_p \pm a|$, and would need to capture the behavior of *cosp u* as *u* varies.

$$cosp\ u = \prod_{n=1}^{\infty} \left(1 - \frac{4u^2}{(2n-1)^2\pi^2}\right), \tag{56}$$

Determine the zeros of *sinp u* and *cosp u*. These zeros are crucial for building the product expansion. Propose a general form of the product based on the identified zeros. For example, for *cosp u*, and *u* are the zeros of *sinp u*, the product might take a form such as.

$$sinp\ u = Ku \prod_{n=1}^{\infty} \left(1 - \frac{u^2}{u_n^2}\right), \tag{57}$$

Now it is needed to determine the constants (like *K*), to ensure the expansion converges correctly and represents *sinp u* and *cosp u*. Also, compare the resulting expansions with known infinite product expansions for standard trigonometric functions to ensure consistency. This approach would lead to a deeper understanding of parabolic functions in relation to their trigonometric counterparts. However, deriving exact forms would require specific knowledge of the zeros of *sinp u* and *cosp u*, and possibly involve advanced techniques in infinite product theory.

Determine the Zeros of sinp u and cosp u:

The zeros of the standard sine function are the values of *x* for which the function *sin*(x) = 0. These occur at all integer multiples of *π*, that is: $x = n\pi$. Larson and Edwards (2013) and Stewart (2016) find that the sine function is defined on the unit circle, where *sin*(*x*) gives the y-coordinate of a point on the circle. Building on the zeros of the standard sine function, it is occurred at; $u = \pi u$, where *n* is an integer. For *sinp u*, we need to find values of *u* that make *sinp u* = 0. This implies $y_p = 0$, which corresponds to points on the parabola where $y_p = 0$. These points are directly on the x-axis. It needs to express these points in terms of the variable *u*.

Similarly, the zeros of the standard cosine function occured at; $u = \frac{(2n-1)\pi}{2}$. For *cosp u*, we need to find values of *u* that make *cosp u* = 0. This also implies $x_p = 0$, which corresponds to points on the parabola where $x_p = 0$. Once the zeros are identified, the infinite product expansion can be formulated. The general structure of an infinite product expansion of parabolic functions, fo *sinp u* and *cosp u*, would be:

$$sinp\ u = Ku \prod_{n=1}^{\infty} \left(1 - \frac{u^2}{u_n^2}\right),$$

and:

$$cosp\, u = K'u \prod_{n=1}^{\infty} \left(1 - \frac{u^2}{v_n^2}\right), \tag{58}$$

Here, u_n represents the zeros of $sinp\, u$ (i.e., where y_p=0), and K is a normalization constant that ensures the correct amplitude. And v_n nrepresents the zeros of *cosp u* (i.e., where x_p = 0), and K' is another normalization constant.

The constants K and K' need to be determined to ensure that the product expansions correctly represent *sinp u* and *cosp u*. This typically involves evaluating the functions at specific points (e.g., u = 0) and comparing them with the expected values. We need to simplify the infinite product expressions using known properties of the parabolic functions, and to identify any patterns or simplifications.

The point P(x_p, y_p) lies on the parabola segment, and subsequently where y_p = 0 corresponds to the x-axis, thus we need to find the conditions under which y_p = 0 in terms of u. For a parabola in its standard form, we get:

$$y_p = k\left(x_p^2 - 2ax_p\right), \text{ setting ; } y_p = 0, \text{ gives:}$$

$$x_p\left(x_p - 2a\right) = 0\,, \tag{59}$$

So, the zeros of y_p occur at:

$$x_p = 0\,, \text{or } \; x_p = 2a\,,$$

However, to link these values to u, we would need to express x_p and y_p as functions of u. Typically, u might represent an angle or another parameter related to the geometry of the parabola. We also need a specific relationship between u and x_p (or y_p) to proceed further.

To proceed further, we need to establish a relationship between the parameter u and the coordinates x_p and y_p of the point P(x_p, y_p) on the parabola. Note that this relationship could be derived from the geometric definition which are using or a parametric representation of the parabola segment.

Therefore, the infinite product expansion for the parabolic function of *cosp u* is:

$$cosp\, u = K'u \prod_{n=1}^{\infty} \left(1 - \frac{u^2}{\left(\frac{(2n-1)\,\pi}{2}\right)^2}\right), \tag{60}$$

To determine K', let's evaluate *cosp u* at u = 0:

$$K' = cosp\, u,$$

Since cosp (0) = 1, it follows that:

$$K' = 1,$$

Thus, the infinite product expansion for *cosp u* becomes:

$$cosp\, u = \prod_{n=1}^{\infty} \left(1 - \frac{u^2}{\left(\frac{(2n-1)\,\pi}{2}\right)^2} \right), \tag{61}$$

These expansions capture the behavior of the parabolic functions *sinp u* and *cosp u* based on their zeros and the angle u related to the coordinates x_p and y_p of point P(x_p, y_p) on the parabola.

Euler's formula (Stewart, 2016), which traditionally relates the trigonometric functions *sinp u* and *cosp u* to the exponential function, can be extended to the parabolic functions *sinp u* and *cosp u*. For parabolic exponential function, we aim to establish a similar relationship:

$$e^{i\sigma_{y_p} u} = sinp\, u \; + i \; cosp\, u \;, \tag{62}$$

Where σ_y is a parameter that used to be determined to match the behavior of parabolic functions, then:

Given the parabolic functions:

$$sinp\, u \; = \frac{y_p}{|x_p \pm a|},$$

$$cosp\, u \; = \frac{x_p}{|x_p \pm a|},$$

Thus, by applying both equations of *sinp u* and *cosp u*, we get a special exponential function of Euler's:

$$e^{i\sigma_{y_p} u} = \left(\frac{y_p}{|x_p \pm a|} + i \frac{x_p}{|x_p \pm a|} \right), \tag{63}$$

To verify or refine this parabolic identity, let us consider the exponential function expanded as a Taylor series:

$$e^{i\sigma_{y_p} u} = \sum_{n=0}^{\infty} \frac{\left(i\sigma_{y_p} u\right)^n}{n!}, \tag{64}$$

Expanding separately for real and imaginary parts, we get:

$$e^{i\sigma_{y_p}u} = \sum_{n=0}^{\infty} \frac{\left(i\sigma_{y_p}u\right)^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{\left(i\sigma_{y_p}u\right)^{2n+1}}{(2n+1)!}, \tag{65}$$

This separates into:

$$sinp\left(\sigma_p u\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\sigma_p u\right)^{2n+1}}{(2n+1)!}, \tag{66}$$

Also:

$$cosp\left(\sigma_p u\right) = \sum_{n=0}^{\infty} \frac{(-1)\left(\sigma_p u\right)^{2n}}{(2n)!}, \tag{67}$$

Now, we will compare these series expansions with the known forms of *sinp u* and *cosp u*. To achieve the parabolic nature:

$$\sigma_p u = \sqrt{\frac{(x_p)^2 + (y_p)^2}{|x_p \pm a|^2}}, \tag{68}$$

Thus, Euler's formula for parabolic functions becomes:

$$e^{i\sqrt{\frac{(x_p)^2+(y_p)^2}{|x_p\pm a|^2}}u} = (sinp\, u + i\, cosp\, u), \tag{69}$$

If we assume a specific normalization, such that σ_p aligns with typical trigonometric functions, we could set $\sigma p = 1$, giving the adapted Euler's formula for both parabolic functions is:

$$e^{iu} = \frac{y_p}{|x_p \pm a|} + i\frac{x_p}{|x_p \pm a|}, \tag{70}$$

$$e^{iu} = sinp\, u + i\, cosp\, u, \tag{71}$$

This formula ties the exponential function directly to the parabolic sine and cosine functions, with the parameter $|x_p \pm a|$ reflecting the geometry of the parabola in relation to the point $P(x_p, y_p)$. This formula relates the exponential function to parabolic analogs of sine and cosine functions, accounting for the specific geometric properties of the parabola.

By the exponential function definition, e^{iu} should vary continuously with x_p, which, as evidenced by the table, clearly changes with respect to the parameter u. In terms of parabolic trigonometric functions, e^{iu} based on formulation (71), where the denominator approaches zero, this formula needs careful handling, suggesting it might require special consideration near the vertex or when x_p aligns with the focus, Table 4.

Table 4: The e^{iu} values for angles u from 0 to π degrees

Angle *u*	*sinp u*	*Cosp u*	the complex function e^{iu}
0	0.80000	2.00000	0.80000+2.00000*i*
15	0.25714	1.37143	0.25714+1.37143*i*
30	0.16000	0.80000	0.16000+0.80000*i*
45	0.11111	0.55556	0.11111+0.55556*i*
60	0.07273	0.90909	0.07273+0.90909*i*
75	0.04667	0.96533	0.04667+0.96533*i*
90	0.04706	0.94118	0.04706+0.94118*i*
105	-0.02353	0.99020	-0.02353+0.99020*i*
120	-0.03478	0.95652	-0.03478+0.95652*i*
135	-0.02262	0.97000	-0.02262+0.97000*i*
150	-0.02759	0.96552	- 0.02759+0.96552*i*
165	-0.01657	0.98573	-0.01657=0.98573*i*
180	-0.02286	0.97143	-0.02286+0.97143*i*

e^{iu} is computed as *sinp u* + *i cosp u*, reflecting the complex exponential function.

By exploring a parabolic version of Euler's famous complex formula, but instead of using ordinary in classical (circular) trigonometry, $\sin u$ and $\cos u$, we're using $sinp\, u$ and $cosp\, u$, which are functions defined relative to a parabola. When we apply this formula across angles *u* from 0 to 180°, table 4 showed that:

- The real part (*sinp u*) varies, sometimes positive and sometimes slightly negative.
- The imaginary part (*i cosp u*) stays positive and close to 1, never becoming negative (even at 180°).
- This behavior is very different from circular trigonometry:
- In circular e^{iu} becomes negative for angles beyond $u = \frac{\pi}{2}$.
- In our parabolic functions, *i cosp u* remains positive even after $u = \frac{\pi}{2}$.

The important point here is that the parabolic "rotation" behaves differently than circular rotation. Even at 180∘, the parabolic cosine is positive. It suggests that motion along the parabola doesn't mirror the traditional cycle of sine and cosine (*where cosine flips sign*). Also, the complex values stay in a specific quadrant. Since *cosp u* is always positive and *sinp u* is mostly small (*sometimes slightly negative*), the points e^{iu} lie mostly near the positive imaginary axis, but shifted slightly left or right.

Regarding the periodicity, results in table 4 indicated that in the circle, e^{iu} goes full circle back to 1 when *u* = 2π. Here, at *u* = 180° (*which is* π), the values don't "return" instead, they stay clustered, showing that the parabolic structure is not truly periodic like a circle. The reason here is that the imaginary part (*cosp u*) is much larger than the real part (*sinp u*) almost everywhere, (see Picture 1 and 2). Results in table 4 indicated that the imaginary component dominates, as the real component is small, even negative sometimes, since no symmetry like in the unit circle . Meaning the "rotation" behavior is dominated by motion along the imaginary (*vertical*) axis. This opens the door to a new type of complex analysis based on parabolic functions rather than circular ones.

This formulation not only reinforces the analogy with Euler's classical identity but also establishes a novel framework for analyzing waveforms, oscillatory behavior, and functional transformations within parabolic geometries. Ultimately, the adapted Euler's formula opens new avenues for advancing the theory of parabolic functions, with promising implications across mathematical physics, signal processing, and the study of geometric transformations. By integrating exponential expressions with parabolic trigonometric analogs, this approach provides a robust mathematical paradigm for characterizing non-circular periodic phenomena inherently linked to parabolic structures.

Conclusions

In this paper, we derive explicit formulas for the series of segments on the parabolic segment, where the focus of the parabola is the center of the origin. It has introduced and explored a novel framework for defining and analyzing trigonometric functions specific to the geometry of the parabola. By extending the classical concepts of sine and cosine, we derived the parabolic trigonometric functions *sinp u* and *cosp u*, tailored to the unique properties of a parabolic curve. These expansions capture the behavior of the parabolic functions *sinp u* and *cosp u* based on their zeros and the angle *u* related to the coordinates x_p and y_p of point P(x_p, y_p) on the parabola. The results presented reveal a fundamental departure from classical circular behavior when extending Euler's formula to parabolic functions. Unlike in circular trigonometry, where the cosine function changes sign and eiu completes a full rotation back to 1 at $u = 2\pi$, the parabolic cosine remains positive even at $u = \pi$, and the complex exponential values e^{iu} cluster predominantly near the positive imaginary axis. This behavior reflects a lack of full periodicity and a dominance of the imaginary component, with the real part remaining small and sometimes negative. As a result, the parabolic "rotation" does not mirror the cyclical symmetry of the circle but instead exhibits a distinct, asymmetric trajectory. These findings emphasize that the parabolic structure introduces a new mode of complex motion, primarily governed by vertical (*imaginary*) movement rather than circular balance between real and imaginary parts. This opens the door to a new branch of complex analysis, tailored to parabolic geometries rather than circular ones. By adapting Euler's classical identity to the parabolic setting, this formulation provides a powerful new framework for analyzing waveforms, oscillatory behaviors, and geometric transformations where parabolic shapes are inherent. The integration of exponential expressions with parabolic trigonometric analogs lays the foundation for broader applications across mathematical physics, signal processing, and the study of non-circular periodic phenomena. Thus, this work establishes a robust paradigm for extending the theory of complex functions into the realm of parabolic structures.

Looking ahead, future research could investigate practical applications of these parabolic functions in areas such as physics, engineering, and computer graphics fields in which parabolic forms and trajectories are frequently encountered. Additionally, extending this analytical framework to encompass other conic sections, such as hyperbolas and ellipses, presents a promising direction for further theoretical development and interdisciplinary exploration.

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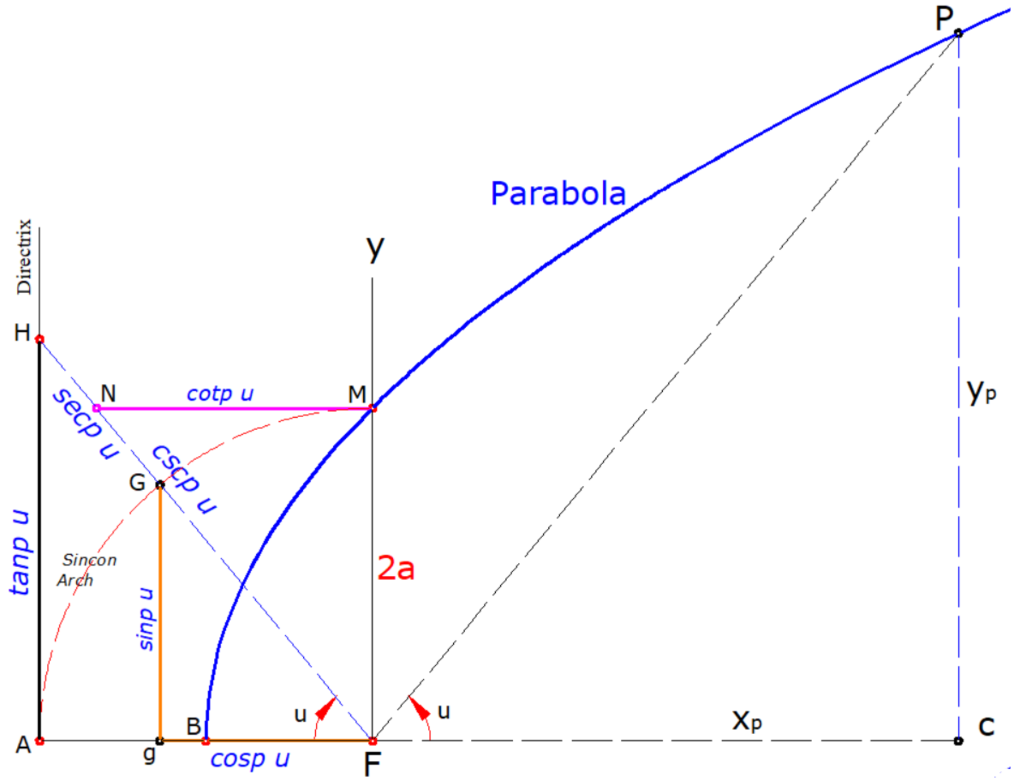
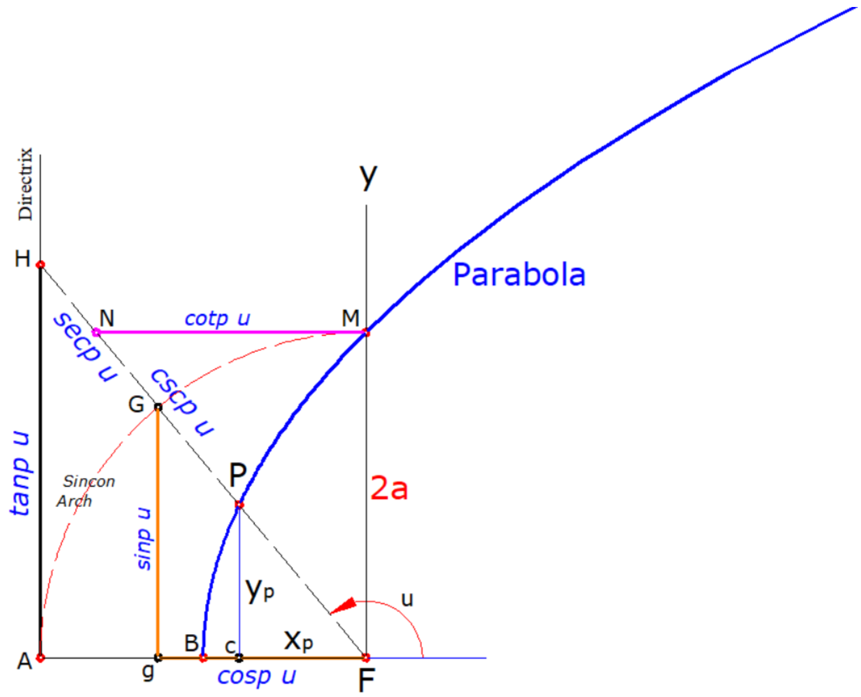
Appendex 1:



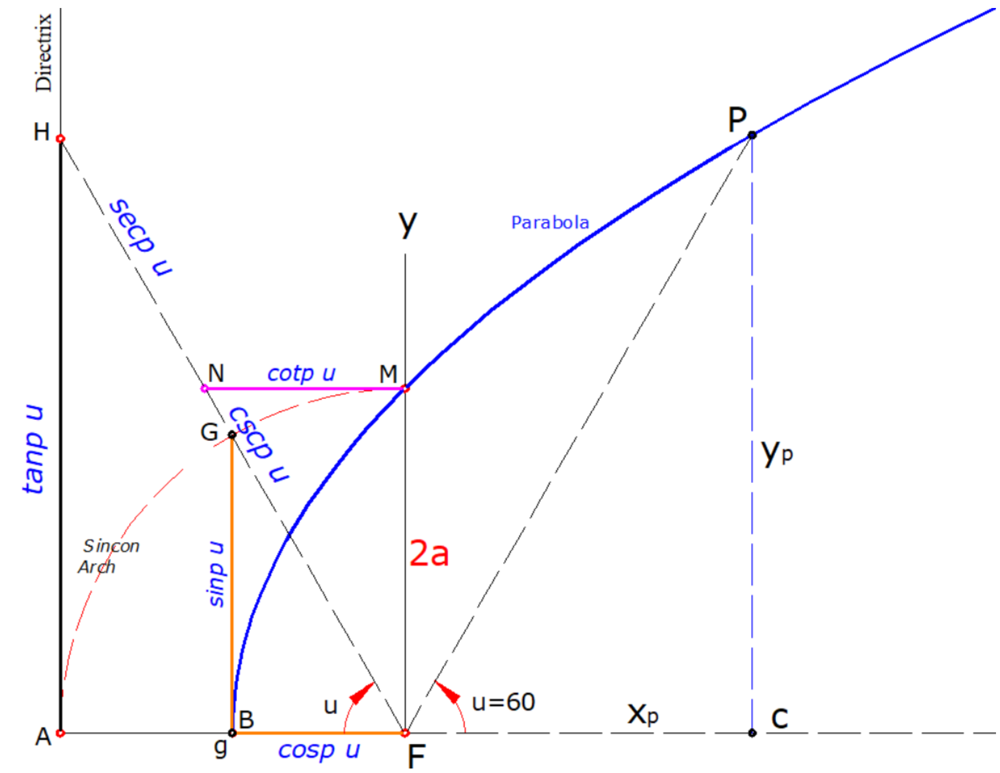
When $(\boldsymbol{u} = \mathbf{50^{º}})$,

Picture A1. The parabolic functions obtained by two ranged angles, (u=120º).



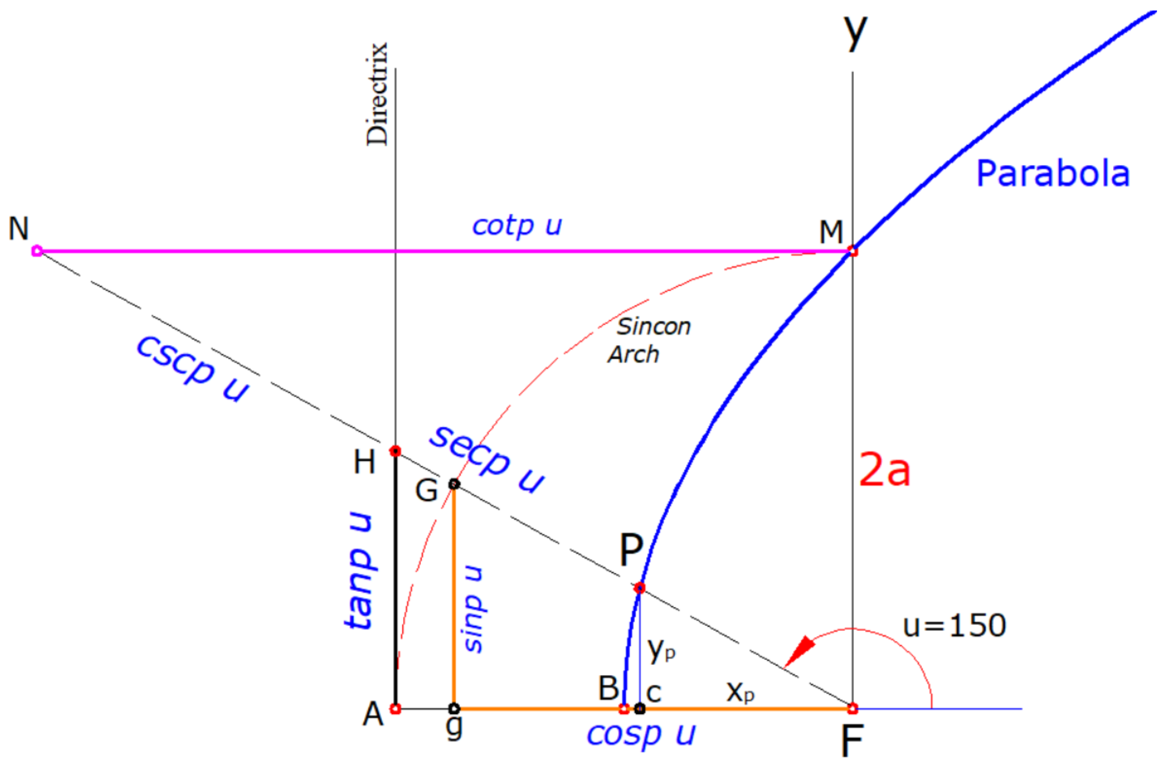
When $(\boldsymbol{u} = \mathbf{120^{º}})$,

Picture A2. The parabolic functions obtained by two ranged angles, (u=120º)



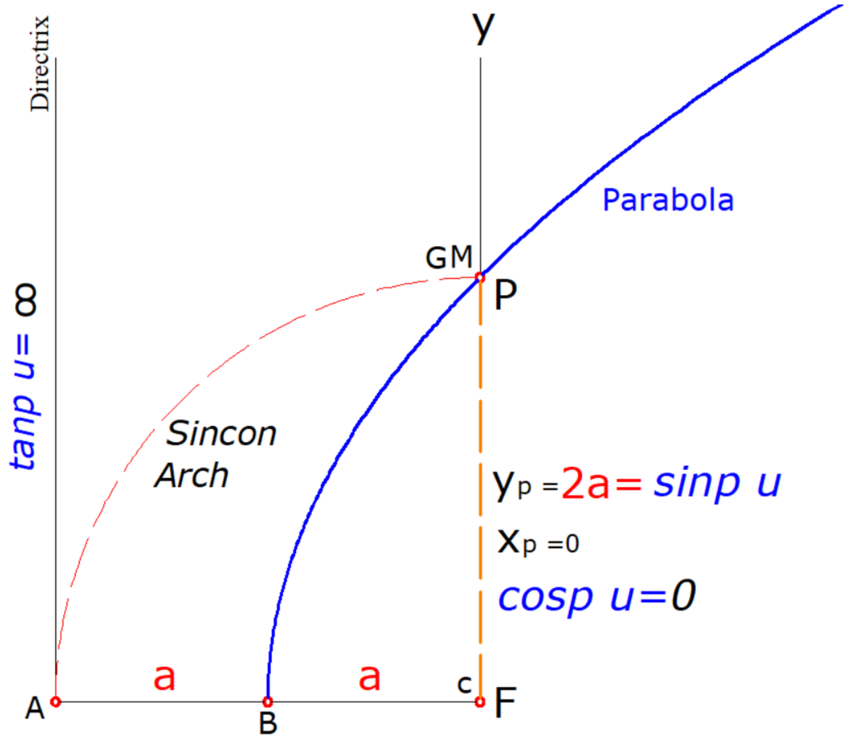
When ($u = 60º$),

Picture A3. The parabolic functions obtained by two ranged angles, (u=60º).



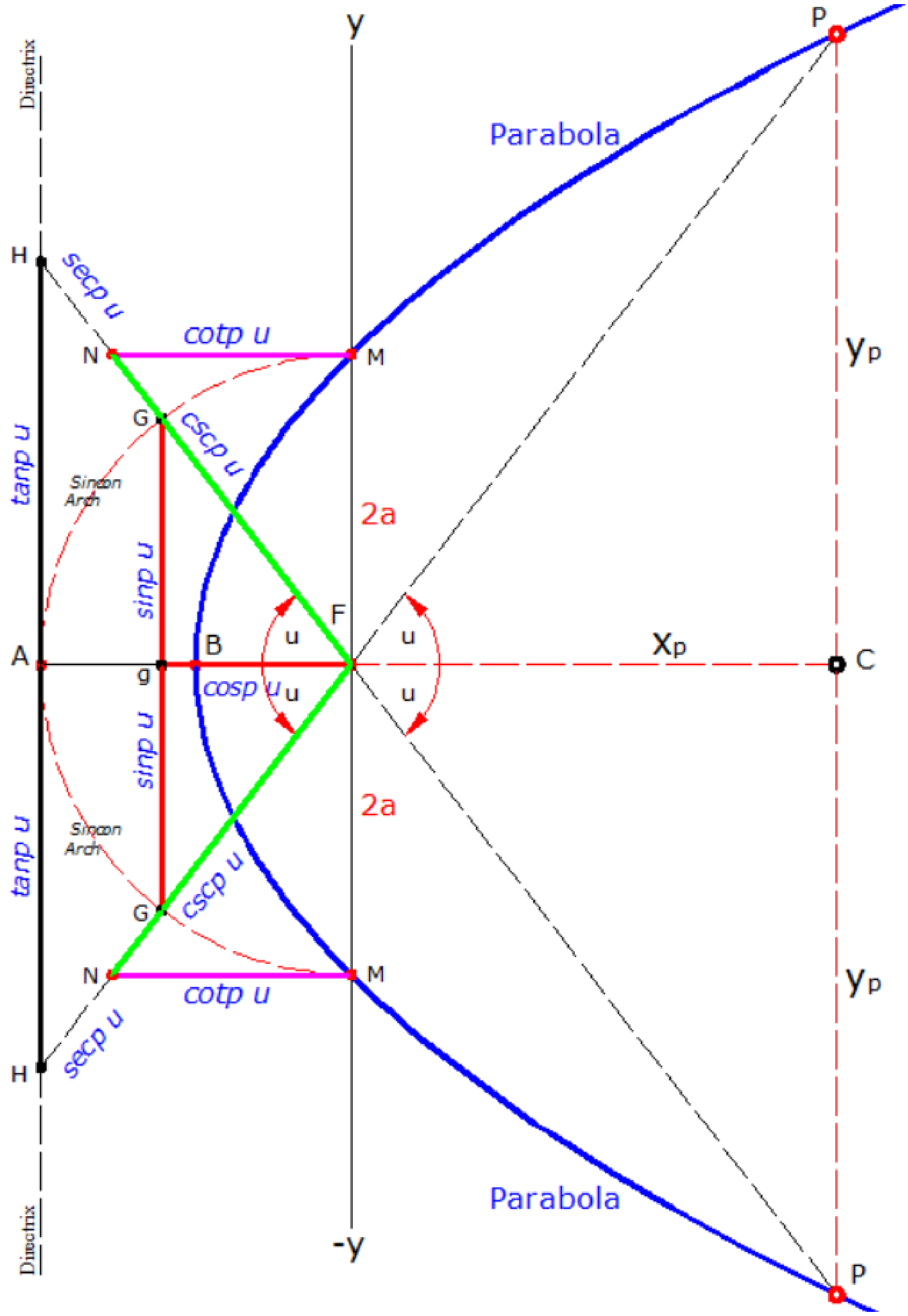
When ($u = 150º$),

Picture A4. The parabolic functions obtained by two ranged angles, (u=150º).



When $\left(u = \frac{\pi}{2}\right)$,

Picture A5. The parabolic functions obtained by two ranged angles, (u=90º).



Picture A6. All the parabolic functions obtained by ranged angles, (u=$\pm$60º)