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Rocket science unveiled: A differential equation exploration of motion

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Abstract:
Through the perspective of differential equations, the report "Rocket Science Unveiled" explores the amazing invention of rocket propulsion. In order to study, comprehend, and forecast the behavior of rocket engines, differential equations are essential. In order to better understand and analyze this intricate anomaly, the report aims to investigate the underlying mathematics of rocket propulsion and how differential equations work. We apply the differential equation to clarify the fuel consumption and thrust generation rates. In addition, we utilize Newton's rule of motion to explain the relationship among thrust, mass, and acceleration. Working on this study allowed us to discover the anticipated outcome for both position location and spacecraft position determination. For iterative operations, we used Euler's approach because the analytical calculation of differential equations is complicated, we used Euler's method for iterative operations. Knowing the rocket's initial or previous value allows us to locate or establish its placements with ease.

Keywords: Exhaust velocity; Propellants; Rocket Propulsion; Thrust Generator.

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Introduction

The history of rocket propulsion is long and diverse, including numerous cultures over several centuries. From the early ninth to the thirteenth centuries, rocketry was utilized in ancient China. Chinese inventors invented "fire arrows," which were simple gunpowder-powered rocket devices used in military and pyrotechnics exhibitions (Capaccioli, 2024). Rocket technology developed westward after the Mongol conquests, which influenced the development of rocket science in the Middle East and Europe (Harvey & Harvey, 2019). The Congreve rocket was developed by British artillery commander Sir William Congreve using Indian rocket manufacturing skills. These rockets saw use in combat in the late 18th and early 19th centuries (Werrett, 2012).

The foundations of modern rocketry were laid in the early 1900s.

Early in the 20th century, visionaries like Robert Goddard in the United States and Konstantin Tsiolkovsky in Russia lay the groundwork for contemporary rocketry (Kotze, 2022). Because of his groundbreaking work in rocket propulsion, Goddard is frequently referred to as the founder of modern rocketry. During World War II, rocketry was developed in a lavish manner. Wernher von Braun's German V-2 rocket became the first long-range guided ballistic missile in history and paved the way for later rocket development. Wernher von Braun was one of the numerous German and American rocket scientists who played a significant role in the advancement of space exploration in the 1940s and 1950s following the war (Shirshekar, 2022). The space era began in 1957 with the Soviet Union's launch of Sputnik 1, the first artificial satellite (Cracknell & Varotsos, 2017).

The Space Race was a name for the intense struggle between the US and the USSR in the 1950s and 1960s (Devezas et al., 2012). Major historical events occurred during this time, such as the first human being in space (Yuri Gagarin in 1961) and the first human lunar landing (Apollo 11 in 1969) thanks to the Apollo program (Spall, 2021). Space exploration has progressed since the Apollo period. Numerous missions have been launched by different nations and space agencies to investigate our solar system and beyond. During this time, reusable launch technologies like the Space Shuttle were developed as well. Private space companies like SpaceX, Blue Origin, and others have grown in prominence in the twenty-first century, modernizing rocket technology and making it more affordable and available. With an emphasis on interplanetary and interstellar travel, space tourism, and the exploration of Mars and other celestial worlds, rocket propulsion technology is still advancing today. The development of rocket propulsion throughout history is evidence of human ingenuity, curiosity, and the will to discover the universe. Simple gunpowder-based rockets have given way to more potent and sophisticated propulsion systems (Serol et al., 2023), enabling previously unheard-of levels of scientific research and space exploration.

Reusable rocket technology has advanced further in the modern era thanks to SpaceX's Falcon 9. SpaceX has completed multiple successful rocket re-flights, lowering the cost of space travel and facilitating more convenient space exploration. Numerous other businesses, including Rocket Lab and Blue Origin, are also developing this reusable technology. Global space organizations and researchers are always developing new ideas and technology to take humanity farther into space (Baum et al., 2022). Concerning the in-depth analysis of the rocket's position and velocity after it leaves the earth's surface, this research is very significant. Because precise space mission predictions are becoming more and more important, we decided to center our research on the rocket's trajectory.

Modeling rocket flight trajectory was the subject of previous work by Sani Abba of the Department of Mathematical Sciences (Computer Science), Faculty of Science, Abubakar Tafawa Balewa University, P.M.B. 0248, Yelwa Campus, Dass Road, Bauchi State, Nigeria (Abba, 2018). While this research focuses more on the position and velocity of the rocket at any given time than it does on problem solving techniques alone, previous research focused on solving numerical problems related to rocket trajectory with a consistent and structured approach. This research advances the notion of easily and conveniently learning the motions of rockets and other spacecraft.

Rocket Propulsion

Rocket Propulsion is the techniques of expelling mass at very high speeds to generate thrust and propel a rocket through space. It is based on several fundamental principles including the use of propellants, nozzles and thrust mechanics (Casiano et al., 2010).

1. Propellants: Propellants are the main components used to generate the high-speed exhaust gases that create thrust. They are typically composed of fuel and oxidizer. Propellants are of different types based on the components they use. They are Liquid propellants (composed of liquid fuel and liquid oxidizer stored in separate chamber and mixed and burned in combustion chamber), Solid propellants (mixture of fuel and oxidizer combined into a solid form), Hybrid propellants (combination of solid and liquid or gaseous oxidizers), Hypergolic Propellants (spontaneously ignite upon contact with each other), Nuclear Thermal Propellants (mainly used for deep-space missions, nuclear thermal propulsion can be used, heating a propellant like hydrogen with a nuclear reactor). The selection of propellants affects rocket performance. Efficiency and thrust generated are influenced by the factors like energy content, combustion temperature, and chemical stability.

2. Nozzles: Rocket nozzles are critical for directing the high-speed exhaust gases generated by the combustion of propellants. The structure of the nozzle accelerates the gas flow to supersonic speeds, increasing exhaust velocity and thrust. The most commonly designed nozzle is the converging-diverging (CD) nozzle. It is divided into three sections: a converging section, a throat and a diverging section. The converging section speeds down the gas and increases its pressure, while the diverging section accelerates the gas to supersonic speeds, increasing its velocity and providing thrust. Depending on factors such as the rocket's altitude, speed, and the specific impulse required, nozzles are designed to operate with supersonic flow to achieve efficient thrust.

3. Thrust Mechanics: Thrust is generated on the account of Newton's third law of motion, which states that for every action, there is an equal and opposite reaction. When the rocket expels mass at high speeds, it generates a force (thrust) in the opposite direction, propelling the rocket forward. The thrust generated by a rocket engine is represented by the thrust equation,

$$T = \dot{m} \, . Ve$$

   Where T is the thrust generated, $\dot{m}$ is the rate of mass flow (i.e. propellant expelled per unit time) and Ve is the effective exhaust velocity.

4. Rocket Equation: The rocket equation describes the change in velocity of the rocket with time due to the expulsion of mass and the resulting thrust. It is based on the principle of conservation of momentum. As a rocket expels mass in one direction, it gains velocity in the opposite direction, on the account of Newton's third law.

The rocket equation is typically expressed in its differential form as:

$$\frac{d(v)}{dt} = \frac{T}{m} - g$$

Where $\frac{d(v)}{dt}$ represents the change in velocity with respect to time, T is the thrust generated, m is the instantaneous mass of the rocket and g is the acceleration due to gravity.

These principles of rocket propulsion, involving the selection of propellants, the structure of efficient nozzles, and the applications of Newton's third law of motion are fundamental to the functioning of all rocket engines.

Differential Equation Governing Rocket Propulsion

We can derive a differential equation that governs the motion of a rocket undergoing propulsion by combining these fundamental principles (Naseri et al., 2022),

$$\frac{d(v)}{dt} = \frac{\dot{m}\,.Ve}{m} - g$$

Where $\frac{\dot{m}\,.Ve}{m}$ represents the thrust-to-mass ratio.

This differential equation shows how the acceleration (rate of change in velocity per unit time) of the rocket is influenced by the rate of mass expulsion, the effective exhaust velocity and the acceleration due to gravity.

Concept of Thermodynamics used in Rocketry:

The thermodynamics describing the combustion process within rocket engines are critical for understanding the behavior of propellants and the generation of thrust. Using differential equations for pressure, temperature, and chemical reactions, these thermodynamic processes can be described as (Wilhelmsen et al., 2017):

1. Conservation of mass (Continuity Equation): The conservation of mass, expressed as a differential equation, describes how the mass of propellants changes during combustion. This equation shows the rate of change of mass with respect to time within the combustion chamber.

$$\frac{dm}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

Where $\frac{dm}{dt}$ is the rate of change of mass, $\dot{m}_{in}$ denotes the mass flow rate of propellants into the combustion chamber, $\dot{m}_{out}$ represents the mass flow rate of exhaust gasses escaping out of the combustion chamber.

2. Conservation of Energy (First Law of Thermodynamics): First law of thermodynamics states that the heat added to the system is equal to the change in

internal energy plus work done by the system. In the context of rocket engines, this equation helps describe the temperature change during combustion.

$$\frac{dU}{dt} = \dot{Q} - \dot{W}$$

Where $\frac{dU}{dt}$ represents rate of change of internal energy, $\dot{Q}$ represents rate of heat transfer into the combustion chamber and $\dot{W}$ represents rate of work done by the combustion chamber.

3. Pressure Changes (Ideal Gas Law): The ideal gas law describes the pressure changes within the combustion chamber by relating pressure, temperature and density in a closed system.

$$PV = nRT$$

Where P is the pressure in the chamber, Vis the volume, n is the number of mole of gas, R is the universal gas constant and T is the absolute temperature.

Research methods

Data gathering

First of all, it contains a range of books and periodicals pertaining to the fundamental research and inventions of Newton. Then it has e-books and a number of websites with the information I wanted regarding my research. It also contains a number of published studies that are connected to it.

Data analysis

Prior to writing this article in chronological sequence, all of the gathered information was thoroughly organized and examined. It was discovered that every piece of data that had been gathered was accurate and free of errors. The argument and evidence support the conclusion, which is not dependent on conjecture.

Results and Discussion

Differential Equations for Rocket Trajectory

1. Newton’s second law: Rocket motion can be described by Newton’s second law (Katsikadelis, 2015), which relates the force applied to the rocket to its mass and acceleration which is expressed as,

$$F = m\frac{d^2x}{dt^2}$$

Where F is the force acting on the rocket, m is the instantaneous mass; x is the position vector and $\frac{d^2x}{dt^2}$ represents acceleration.

2. Force of Gravitation: Rocket trajectory is greatly affected by the gravitational force (Chen & Xia, 2016). It follows Newton's law of universal gravitation,

$$F_{gravity} = -\frac{GM_e m}{r^2}\hat{r}$$

Where:
$F_{gravity}$ represents gravitational force.
G is the universal Gravitational constant.
M_e denote the mass of the Earth.
m is the mass of the rocket.
r is the distance between center of the Earth and rocket.
$\hat{r}$ is the unit vector in the direction of r.

3. Equations of Motion: The basic equation of motion is based on Newton's second law and is given by F = m.a. for a rocket (Nakayama, 2018), accounting for gravitational force $F_{gravity}$ and thrust F_{thrust}, the equation is represented as,

$$m\frac{d^2x}{dt^2} = F_{thrust} - F_{gravity}$$

Solving these differential equations analytically is much complex. So Numerical methods such as Euler's method are used to solve these equations:

Euler's method involves approximating the solution by making small steps using derivatives (Coutand & Shkoller, 2012). For a differential equation like the above, the solution can be found by:

$$x(t+\Delta t) = x(t) + v(t).\Delta t$$
$$v(t+\Delta t) = v(t) + a(t).\Delta t$$

Here, x(t) is position, v(t) is velocity, a(t) is acceleration and Δt is the time step.

Illustration

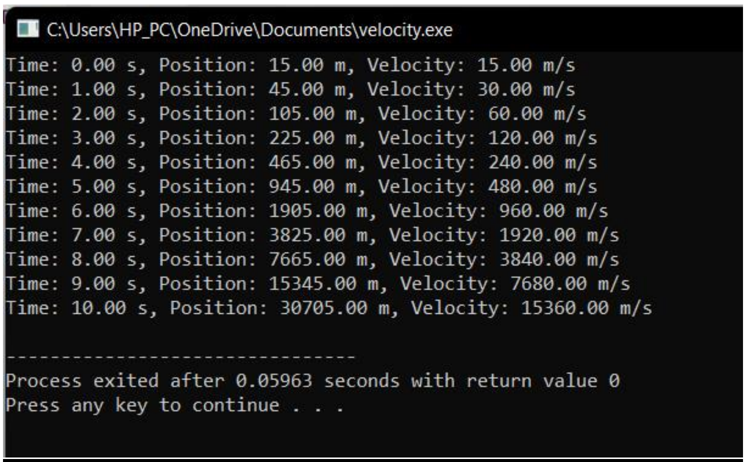
Let us assume the National space agency of Nepal "HOMOS" is launching its1000 kg weighted rocket to relay its public communication satellite 'LINK-I' in order to provide free 4-G network signals throughout the country. The propellants used in the rocket produces 15000 N of thrust. Position and Velocity of the rocket at each time step (i.e. 1second) can be demonstrated as.

Limitations

a. It is the basic representation and does not account for many real-world complexities like air resistance, changing mass or complex thrust profiles.
b. It is a simplified model and doesn't cover all aspects of rocket dynamics.
c. The higher calculation may violate the existed laws of relativity as it is simplified model design for educational purposes.

Here, is a code written in C programming language to calculate position and velocity of rocket of mass 1000 kg from its starting phase to 10 seconds of its launch,

```
#include <stdio.h>
#define dt 1 // Time step
// Function to calculate acceleration
float acceleration(float force, float mass)
 {
   float g = 9.81; // Acceleration due to gravity (m/s^2)
   float acc = force / mass ; // Newton's second law
   return acc;
}
int main()
{
   float mass = 1000; // Initial mass of the rocket (kg)
   float thrust = 15000; // Thrust of the rocket (N)
   float time = 0;
   float velocity = 0;
   float position = 0;
  while (time <= 10)
        {
          // Simulation time
      float force = thrust;          // For simplicity, assume constant thrust
      float acc = acceleration(force, mass);
      velocity = velocity + acc * dt;
      position = position + velocity * dt;
      mass = (thrust / velocity) * dt;          // Simplified mass decrease due to propulsion
      printf("Time: %.2f s, Position: %.2f m, Velocity: %.2f m/s\n", time, position,
velocity);
      time += dt;
   }
   return 0;
}
```



Picture 1. Output of code written in C programming language

In the same way, we can easily obtain velocities and positions of the rocket at different time intervals. The higher calculation may find that the velocitiy of rocket exceeds the speed of light which is due to our assumption of constant thrust . Furthermore we have neglected the air resistance and changing mass of the rocket due to combustion of fuel.

Conclusion

This project seeks to unveil the underlying mathematics and physics that govern rocket dynamics emphasizing the indispensable role of differential equations. By exploring the differential equation models used in rocket science, this project aims to provide a comprehensive understanding of the complexities involved in determining positions and velocities of rockets/satellites.

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