Alifmatika: Jurnal Pendidikan dan Pembelajaran Matematika
Volume 5, Issue 1, 1-20, June 2023
e-ISSN: 2715-6109 | p-ISSN: 2715-6095
https://journal.ibrahimy.ac.id/index.php/Alifmatika

An elementary treatise on elliptic functions as trigonometry

Laith H. M. Al-ossmi*
College of Engineering, Thi-qar University, Al-Nasiriya City 370001, Iraq
*hardmanquanny@gmail.com

Received: February 5, 2023 | Revised: March 3, 2023 | Accepted: May 10, 2023 | Published: June 15, 2023
*Corresponding author

Abstract:
This article concerns the examination of trigonometric identities from an elliptic perspective. The treatment of elliptic functions presented herein adheres to a structure analogous to the traditional exposition of trigonometric functions, with the exception that an ellipse replaces the unit circle. The degree of similarity between the elliptic functions and their trigonometric counterparts is moderated by the periodicity of the so-called El- functions. These identities not only establish the values of the functions, but also establish a correlation between their ratios and the major and minor axes of the underlying ellipse. The resemblance between the functions is somewhat modified by the periodic nature of the El-identities, whereby each ratio is associated with the major and minor axis of the ellipse. This article adopts the notation (E) to denote the El- functions and distinguish them from the opposite circular functions.

Keywords: Elliptic functions, Circular functions, Geometric characteristics, Trigonometric identifies.

How to Cite: Al-ossmi, L. H. M. (2023). An elementary treatise on elliptic functions as trigonometry. *Alifmatika: Jurnal Pendidikan dan Pembelajaran Matematika*, 5(1), 1-20. https://doi.org/10.35316/alifmatika.2023.v5i1.1-20

Introduction

Within the realm of mathematics, conic sections are defined as a collection of curves derived from the intersection of a plane with the surface of a right cone (Boyer & Merzbach, 1968; Byrd & Friedman, 1971). While the ellipse is one of three types of conic sections, historically and in Euclidean geometry, it has been extensively studied and has yielded a plethora of interesting results. The properties of the ellipse were systematically examined by Apollonius of Perga in 200 BC (Hazewinkel, 2001; Smart & Schwandt, 2012), and the Persian mathematicians Omar Khayyám and Avicenna employed conic sections to solve algebraic equations of degree no higher than three (Schaum, 2009; Liu et al., 2001; Kim & Kim, 2021; Yang et al., 2021).

In analytic geometry, an ellipse can be defined as a set of points that satisfy a quadratic equation in two variables, and can be described as a plane algebraic curve of degree two with various geometrical properties (Turner, 2010; Liu et al., 2001; Shen et al., 2006; Salas et al., 2022; Dimitrov, 2022). The major axis of the ellipse is the chord between the two vertices, whereas the minor axis (a) is the shortest diameter and the major axis (b) is the longest diameter. This property can be utilized to generate the equation that the points on the ellipse satisfy, which can be expressed in the standard form of an ellipse equation.

An ellipse is a curve, and hence the notion of taking the sine or cosine of it is not well-defined (W. Wang et al., 2015). However, an ellipse with semi-major axis (a) and semi-minor axis (b) can be parameterized by the equations x = a cos(t) and y = b sin(t), where t is the angle of the ray from the ellipse center that ranges from 0 to 2π. Additionally, the corresponding functions for the ellipse, $x^2/a^2 + y^2/b^2 = 1$, are merely scalar multiples of the circular functions: the coordinates of the ellipse are parameterized by (a cos t, b sin t), so the elliptical functions would just be (a cos) and (a/b) tan. Thus, an ellipse can be described by a segment ray originating from the ellipse center and extending to its circumference. The ellipse can be characterized by three parameters: a, b, and u. The ratio between the major and minor axis of the ellipse determines its ellipsoidal properties, while the value of the angle is determined by the position of the point on the ellipse perimeter that is constructed from the ellipse center.

Trigonometry functions are mathematical functions that relate the angles of a triangle to the lengths of its sides (Altman & Kidron, 2016). The six basic trigonometry functions are sine, cosine, tangent, cosecant, secant, and cotangent (Siyepu, 2015). Trigonometry identities, on the other hand, are mathematical equations that relate the trigonometry functions to one another. These identities are derived from the fundamental relationships between the angles and sides of a right triangle, and they are used to simplify trigonometry expressions and solve trigonometry problems. This geometric fact has led to the development of a set of elliptic functions in this article, derived from these parameters, to deal with trigonometric functions. The obtained identities for an ellipse and a unit circle are applicable to circular identities but are modified to account for the properties of the ellipse in a two-dimensional plane.

Research Methods

This paper presents a geometric technique for constructing a set of trigonometric identities derived from the properties of ellipses. The objective of this work is to explore the connections between conic sections, specifically circles and ellipses, through the lens of trigonometry. The derived identities include combined functions relating to the ellipse and its corresponding circle, using the major and minor axes as radii. Two major tasks are undertaken in this paper. Firstly, these identities are analyzed using a unit circle centred at the same point as the ellipse. Secondly, two cases are considered, where the minor axis (a) and major axis (b) of the ellipse are used as the radius of the unit circle. The definitions of trigonometric functions are shared between the ellipse and the circle, where sine (sin u) and cosine (cos u) are similarly defined with respect to the circle's radius. The angle of arc length is measured from the positive x-axis in both cases. The El-functions notation, including Esin (u), Ecos (u), Etan (u), etc., is introduced to represent ellipse ratios. These identities are named with the letter (E) to distinguish them from the circular functions used for periodic notation. Additionally, this paper does not cover Jacobi elliptic functions or Weierstrass elliptic functions, which are practical notions of complex analysis and expansion methods for periodic wave solutions of ellipses (Q. Liu & Gao, 2021; Li et al., 2021).

One more important peculiarity of the approach in this paper should be mentioned. This paper does not deal with the Jacobi elliptic functions (Alquran & Jarrah, 2019; Dragović & Radnović, 2011), or the Weierstrass elliptic functions (Bialy, 2022;

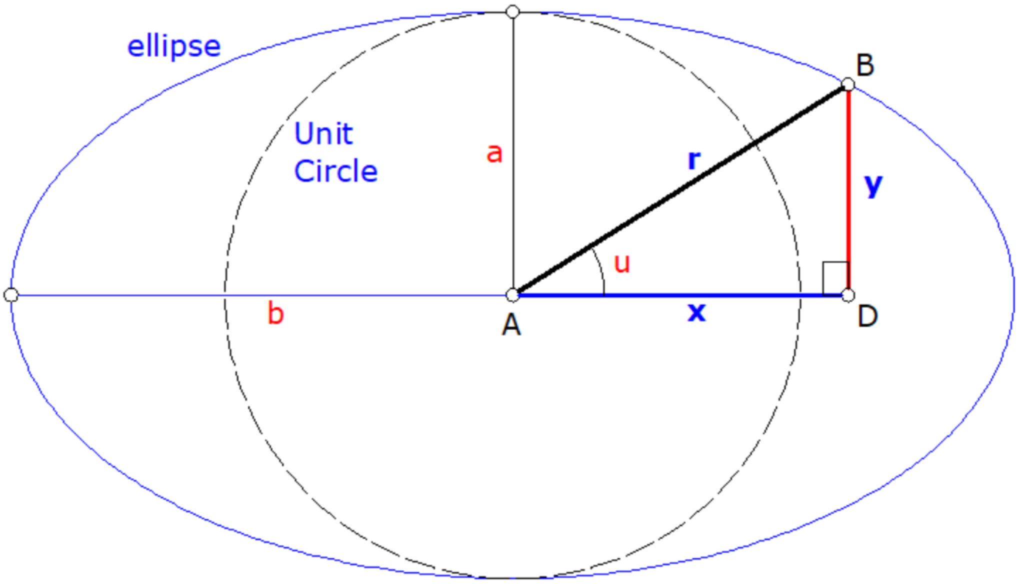
Reznik et al., 2021), which refer to practical notions of complex analysis, in expansion method and periodic wave solutions the of ellipse, in particular (Liu et al., 2001; Wang & Jin, 2022). The conic functions derived in this paper are referred to as elliptic identities in trigonometry. Although the Jacobian and Weierstrass elliptic functions are special cases of such general elliptic functions, they are outside the scope of this work. This article represents an advanced step in studying elliptical forms and combined functions. The article adopts the notation (E) to denote the El- functions and distinguish them from the opposite circular functions. This work is an advanced step added to these dealt with elliptic forms and these combined functions (L. Al-ossmi, 2023; L. H. M. Al-ossmi, 2022).

Research Result

Definition of Elliptic and Trigonometric Functions

The present article provides a definition of elliptic and trigonometric identities. An ellipse can be described as a stretched circle whose locus of points is determined by the sum of distances from two foci that equals a fixed value. While trigonometric functions are defined with respect to a circle, elliptic functions are a generalization encompassing other conic sections, particularly the ellipse. To establish a parallel with normal trigonometry, defining functions that connect to the unit circle is useful. In the case of ellipses, the coordinates of any point on the perimeter are normalized concerning the ellipse's major and minor axis.

These elliptic functions are expressed in two variables, as they depend on the modulus that defines the shape of the ellipse. The relation to trigonometric functions is incorporated in the notation of the ellipse proportions, denoted by the variables x and y for a point on the ellipse and a and b for the ellipse's major and minor axis, respectively. The present study examines elliptic identities in two cases: the first is when the radius of the unit circle is the major axis of the ellipse (b), and the second is when the radius is the minor axis of the ellipse (a). Let B(x,y) be a point on an ellipse, and let (AB) be the line segment from the ellipse center (A) to the ellipse circumference. The angle of the segment on the minor axis is represented by u, where $0 \geq u \leq 2\pi$. Then, the coordinates of point B are determined by the vertical segment from point D on the minor axis (BD) (see Picture 1).



Picture 1. The Pythagorean formula for Esines and Ecosines, where the minor axis of the ellipse is the radius of the Unit Circle, (a)

However, the values of x and y are intricately related to the arc length u of the ellipse. The coordinates of point ordinates, which are line segments on the circumference of the circle measuring arc length, are defined to be cos u and sin u, respectively, thereby simplifying the circular functions. This paper does not delve into the complex variable theory of elliptic functions. Instead, it uses geometrical definitions to establish addition formulas for the ellipse curvature and all other trigonometric identities. The study of right-angled triangles from the center point of the ellipse is the foundation of the theory, which generalizes to arbitrary triangles with its circumference. While the development of this paper is a part of trigonometric functions, the theory deals with ellipses rather than circles. We define the elliptic ratios, or El-functions, for an acute angle by drawing a right-angled triangle, one of whose acute angles is u. The subject matter is more like a branch of geometry than a fundamentally new concept. From a mathematical standpoint, all El- functions can be derived using the concept of similar triangles applied to an ellipse, where the x- and y-coordinates of points on the unit ellipse serve as a generalization of the relations for the coordinates of points on the unit circle.

Elliptic Identities (El- Functions)

The present study concerns the investigation of a new set of functions that can be obtained by drawing a given ellipse and a unit circle with a shared center point, where the major axis of the ellipse (b) is equal to the radius of the circle (r). These newly produced functions, termed as elliptic ratios, are derived from the properties of the ellipse and are denoted by names ending with the letter (E), collectively referred to as El- functions. The set of El- functions includes Esin (u), Ecos(u), Etan (u), Esec (u), Ecsc (u), and Ecot (u), which are presented in Table 1.

Table 1: El- functions associated with these circular functions

Circular	*sin u*	*cos u*	*tan u*	*sec u*	*csc u*	*cot u*
Elliptical	*Esin u*	*Ecos u*	*Etan u*	*Esec u*	*Ecsc u*	*Ecot u*

From a mathematical perspective, the El-functions can be derived using the concept of similar triangles that pertain to an ellipse. The x- and y-coordinates of points on the unit ellipse can be considered a generalization of the relations for the coordinates of points on the unit circle. When considered as functions of a complex variable, deep methods of the elliptic functions will not be discussed in this paper. By utilizing these geometrical definitions, addition formulas can be established for particular curvatures, which, in this case, is the ellipse (and hence all other trigonometric identities). Subsequently, the circular functions are studied rigorously to obtain new fundamental formulas of elliptic identities.

Therefore, the theory begins by studying right-angled triangles from the center point of the ellipse and then generalizes to arbitrary triangles with its circumference. This paper examines the El-functions in two cases relating to the unit circle's radius and the ellipse axis, a and b, associated with these circular functions.

In the first case, the unit circle's radius is the minor axis of the ellipse (a):

In this case, the unit circle with a radius of (a) shares the center point of the ellipse, denoted as (A), and the ray constructed from the center point to a point (B) on the ellipse perimeter is the ellipse radius (r). The vertical and horizontal projections of the point B are the y- and x-coordinates, respectively, as depicted in Figure 2. According to the ellipse proportions, the notation of these three functions of the ellipse can be explained concerning the point B (x, y) as follows.

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1\ ,$$

$$\frac{x^2}{b^2} = 1 - \left(\frac{y}{a}\right)^2,$$

$$x^2 = b^2 - b^2\left(\frac{y}{a}\right)^2,$$

$$x = \sqrt{b^2\left(1 - \frac{y^2}{a^2}\right)}\ , \tag{1}$$

$$y = \sqrt{a^2\left(1 - \frac{x^2}{b^2}\right)}\ , \tag{2}$$

In order to derive the El-functions, the right triangle (BAD) with sides equal to the ellipse radius, r, and the vertical and horizontal projections from a point on the ellipse, (BD) and (AD), are utilized. This construction is illustrated in figure 2 and serves as a crucial step in establishing the El-functions, which are considered to be one of the most significant trigonometric identities.

$$AD = x, \text{ and } BD = y, \text{ then:}$$

$$Esin\, u = \left(\frac{y}{r}\right), \tag{3}$$

$$Ecos\, u = \left(\frac{x}{r}\right), \tag{4}$$

$$Etan\, u = \left(\frac{y}{x}\right), \tag{5}$$

After considering all parameters, including a, b, x, y, and r, the values of the three essential El-functions, namely $Esin$, $Ecos$, and $Etan$, can be expressed as follows.

$$Esin\, u = \frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{r}, \tag{6}$$

$$Ecos\, u = \frac{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}{r}, \tag{7}$$

$$Etan\, u = \frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}, \tag{8}$$

Using the Pythagorean theorem, the value of the radius, r, which is the hypotenuse of the right triangle with angle (u), can be employed to determine the El-functions, $Esin$ and $Ecos$, as follows.

$$r = \frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{Esin\, u}, \tag{9}$$

$$r = \frac{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}{Ecos\, u}, \tag{10}$$

Using the equations 9 and 10, we can express the E-identities in terms of Esine and Ecosine by applying basic trigonometric identities.

$$\frac{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}{Ecos\, u} = \frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{Esin\, u}, \tag{11}$$

$$Esin\,u = \left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}\right)Ecos\,u, \tag{12}$$

$$Ecos\,u = \left(\frac{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}\right)Esin\,u, \tag{13}$$

Likewise, the E-identities provide a way to represent trigonometric functions in terms of their complements. Specifically, each of the six El-functions is equivalent to its co-function assessed at the complementary angle of (u).

$$Esec\,u = \left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}\right)Esin\,u, \tag{14}$$

$$Ecsc\,u = a\left(\frac{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}\right)Ecos\,u, \tag{15}$$

It should be noted that there exist additional El-functions, although they may not be as significant as the ones mentioned previously. Nonetheless, these identities can be derived from the fundamental El-functions described earlier.

$$Esin\,u = \left(\frac{y}{x}\right)Ecos\,u, \tag{16}$$

$$Ecos\,u = \left(\frac{x}{y}\right)Esin\,u, \tag{17}$$

$$Esec\,u = \left(\frac{y}{x}\right)Esin\,u, \tag{18}$$

$$Ecsc\,u = a\left(\frac{x}{y}\right)Ecos\,u, \tag{19}$$

And from equation 3, the *Ecot* can be reformed related to *Etan*;

$$Ecot\,u = a\left(\frac{x}{y}\right) = \frac{a}{Etan\,u}, \tag{20}$$

And from equations 1 and 2, the reversed fomes of *Ecsc* and *Esec* can be reformed as;

$$Ecsc\,u = a\left(\frac{r}{y}\right) = \frac{a}{Esin\,u}, \quad (21)$$

$$Esec\,u = \left(\frac{r}{x}\right) = \frac{1}{Ecos\,u}, \quad (22)$$

The El functions can be expressed in terms of the trigonometric functions, as shown in equations 9, 10, and 11. Moreover, the El-functions can also be derived from the trigonometric addition formulas by considering the right triangle (DAB) with sides r, x, and y and using the definitions of Esine and Ecosine in terms of these sides. Therefore, there is a close relationship between E- functions and trigonometry; the former can be derived from the latter or expressed in terms of it.

$$\frac{Etan\,u}{a\,sin\,u} = \left(\frac{r}{a}\right), \quad (23)$$

$$Etan\,u = r\,sin\,u\,, \quad (24)$$

$$\left(\frac{y}{x}\right) = r\,sin\,u\,, \quad (25)$$

$$sin\,u = \frac{y}{x\,r}, \quad (26)$$

$$sin\,u = \frac{y}{r\sqrt{b^2\,(1-\frac{y^2}{a^2})}}, \quad (27)$$

$$sin\,u = \frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{r\,x}, \quad (28)$$

$$sin\,u = \frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{r\sqrt{b^2\,(1-\frac{y^2}{a^2})}}, \quad (29)$$

Consequently, they can also be derived from the ratios with trigonometric functions as shown above (see Figure 1).

$$\frac{x}{cos\,u} = \left(\frac{r}{a}\right),$$

$$cos\,u = a\left(\frac{x}{r}\right), \quad (30)$$

$$cos\,u = a\,Ecos\,u, \quad (31)$$

$$Ecos\,u = \frac{cos\,u}{a}, \quad (32)$$

$$cos\,u = a\left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{r}\right), \tag{33}$$

$$cos\,u \;= a\,Ecos\,u, \tag{34}$$

$$cos\,u \;= \left(\frac{a}{Esec\,u}\right), \tag{35}$$

$$Esec\,u \;= \frac{a}{cos\,u}, \tag{36}$$

$$Esec\,u \;= \frac{a}{\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{r}} = \frac{a\,r}{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}, \tag{37}$$

$$Esec\,u \;= \frac{a\,r}{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}, \tag{38}$$

$$Esin\,u \;= \left(\frac{x}{y}\right)\frac{a\,r}{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}, \tag{39}$$

$$Ecsc\,u \;= a\left(\frac{x}{y}\right)\frac{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}{r}, \tag{40}$$

In a similar manner, obtaining the derivative of each term provides supplementary series for Etangent, tangent to Ecot and cot, which fulfil the given angle formulas, such as:

$$\frac{Etan\,u}{tan\,u} = \left(\frac{b}{a}\right),$$

$$tan\,u = \left(\frac{a}{b}\right)\,Etan\,u\,, \tag{41}$$

$$tan\,u = \left(\frac{a}{b}\right)\left(\frac{y}{x}\right), \tag{42}$$

$$tan\,u = \left(\frac{a}{b}\right)\left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}\right), \tag{43}$$

$$tan\,u = \left(\frac{a}{b}\right)\left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{x}\right), \tag{44}$$

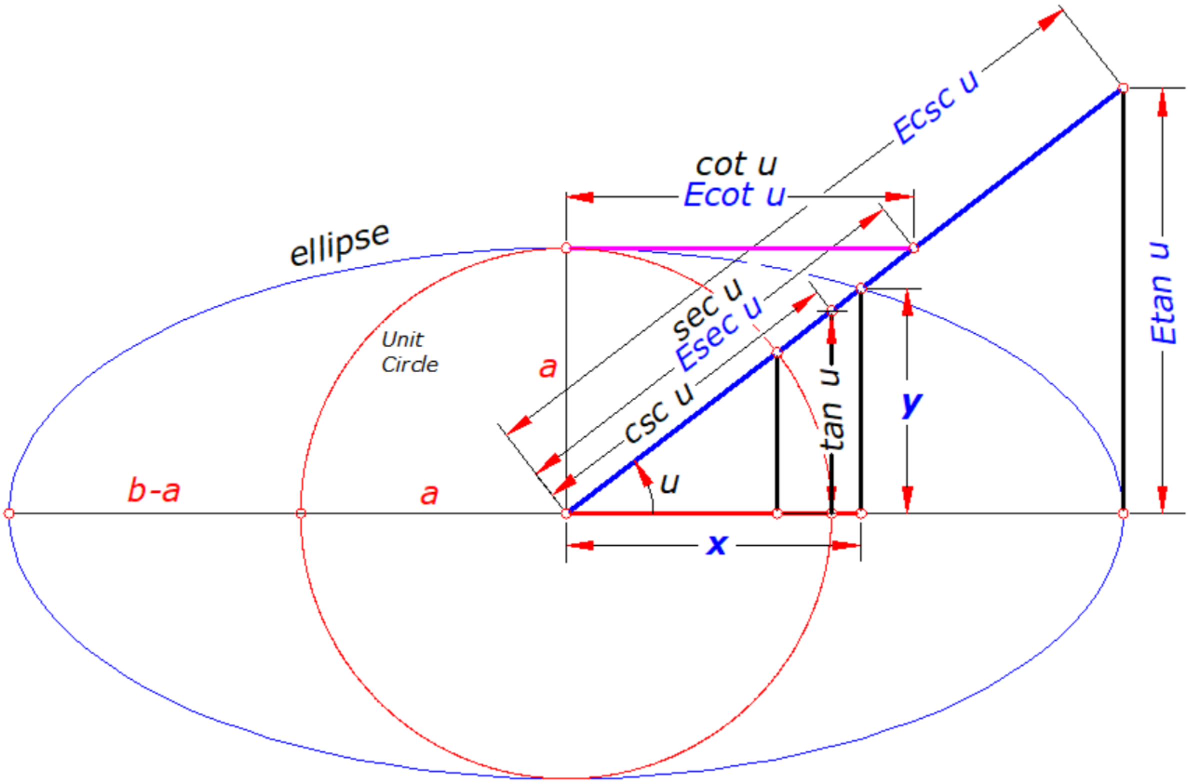
$$tan\ u = \left(\frac{a}{b}\right)\left(\frac{y}{\sqrt{b^2\ (1-\frac{y^2}{a^2})}}\right), \tag{45}$$

$$cot\ u = \frac{1}{tan\ u} = \left(\frac{b}{a}\right)\left(\frac{x}{y}\right), \tag{46}$$

$$cot\ u = \left(\frac{b}{a}\right)\left(\frac{\sqrt{a^2\ (1-\frac{x^2}{b^2})}}{y}\right), \tag{47}$$

$$cot\ u = \left(\frac{b}{a}\right)\left(\frac{x}{\sqrt{b^2\ (1-\frac{y^2}{a^2})}}\right), \tag{48}$$

The above equations indicate that these elliptical functions are forms of trigonometry related to ellipses' properties, as all of them can be expressed in terms of trigonometric identities. The definitions of Esine and Ecosine express them as infinite series or as solutions to certain differential equations, enabling their extension to arbitrary positive and negative values and even to complex numbers. Consequently, Esine and Ecosine are represented using functional notation with the abbreviations Esin and Ecos, and their argument, u, is typically expressed in radians or degrees. When the argument is simple enough, the function value is often written without parentheses as Esin u rather than as Esin(u). This article assumes that the angle is measured in radians, as depicted in Picture 2. These periodic functions should be qualitatively similar to the sine and cosine functions, particularly for small eccentricities.



Picture 2. For any angle (u), plot the E- identities compared with trigonometry functions using the first case; where the minor axis of the ellipse is the radius of the Unit Circle, (a)

It has been observed that when (u = π/4), the functions have special relationships indicating that the ellipse is aligned with the unit circle, and the major axis (b) is equal to the circle radius (r). When (u=π/4), the E-formulas can be used to derive the length of the ray (r) for any ellipse, and it can be expressed by providing the values of:

$$(Esin\ u = Ecos\ u).$$

In the case where (u=π/4), there are special relationships between the E-functions and trigonometric functions. This occurs when the ellipse is centered with the unit circle and the major axis (b) is equal to the radius (r) of the circle. Furthermore, when (u=π/4), the El-formulas can be restructured to determine the length of the ray (r) for any ellipse.

$$Esin\ u = Ecos\ u\ ,$$

$$Etan\ u = tan\ u,$$

$$Esec\ u = \csc u = \sec u,$$

$$Ecot\ u = a,$$

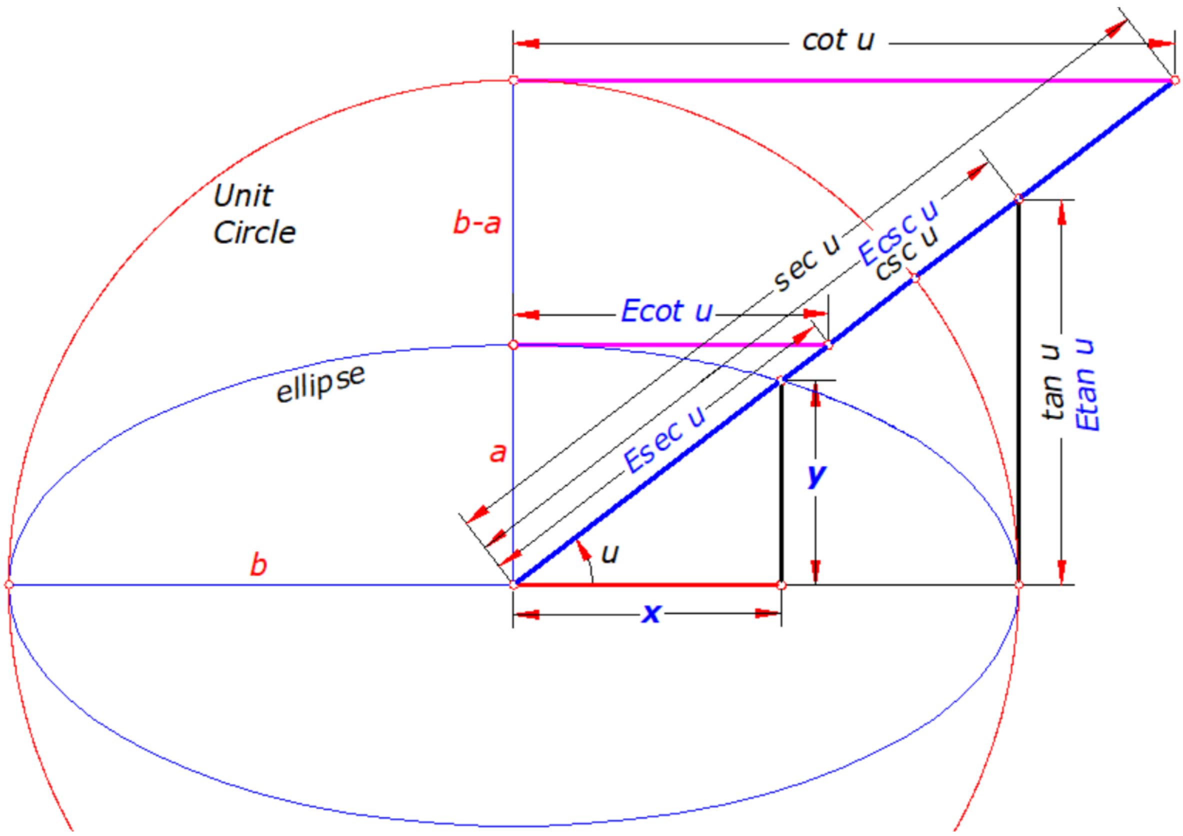
Here, if $\left(u = \frac{\pi}{4}\right)$, then ($sin\, u$ and $cos\, u$) functions represent the opposite forms of El-functions, both of which are used for any ellipse of (a) and (b).

The second case: When the unit circle's radius is the major axis of the ellipse, (b).

In this case, the El-functions are still determined by the same parameters a, b, x, y, and r, but the important point is that the value of Etangent is equal to the tangent. Therefore, we have the following relationships for any set of ellipses sharing a center point and the value of the major axis b when u=π/4:

$$Etan\ u = tan\ u = \frac{\sin u}{\cos u}, \tag{49}$$

Picture 3. definitions relations for Etanget, Ecotanget, Esecant, and Ecosecant in terms of *sine* and *cosine* where the major axis of the ellipse is the radius of the Unit Circle, (b).



Picture 3. for any angle (u), plot the El-functions compared with trigonometry functions using the second case; where the major axis of the ellipse is the radius of the Unit Circle, (b)

In this case, Etan u is equivalent to both Ecot u and cot u, which are reciprocal functions of the tangent function. Therefore, the relationship between sin, cos, and Etan u can also be expressed in terms of Ecot u and cot u, depending on which reciprocal function is preferred for the given problem or context.

$$\frac{Ecot\,u}{cot\,u} = \left(\frac{b}{a}\right),$$

$$Ecot\,u = cot\,u\left(\frac{b}{a}\right), \tag{50}$$

$$cot\,u = Ecot\,u\left(\frac{a}{b}\right), \tag{51}$$

$$\frac{1}{tan\,u} = Ecot\,u\left(\frac{a}{b}\right), \tag{52}$$

$$Ecot\,u = \frac{1}{tan\,u}\left(\frac{a}{b}\right), \tag{53}$$

$$Ecot\,u = \frac{1}{Etan\,u}\left(\frac{a}{b}\right), \tag{54}$$

$$Ecot\,u = \frac{Esin\,u}{Ecos\,u}\left(\frac{a}{b}\right), \tag{55}$$

$$Ecot\,u = \left(\frac{y}{x}\right)\left(\frac{a}{b}\right), \tag{56}$$

$$Ecot\,u = \left(\frac{a}{b}\right)\left(\frac{y}{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}\right), \tag{57}$$

$$Ecot\,u = \left(\frac{a}{b}\right)\left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{x}\right), \tag{58}$$

Therefore, since Etan u = tan u (as stated in equation 49), a connection can be established between sin, cos, and Etan u with the major axis of the ellipse (which is equivalent to the radius of the unit circle, b) based on the right triangle BDA. This relationship can be expressed as follows.

$$\frac{Etan\,u}{b\,sin\,u} = \frac{b}{cos\,u},$$

$$Etan\,u = \frac{1}{cos\,u}\left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{r}\right), \tag{59}$$

$$Etan\,u \;= sec\,u\left(\frac{\sqrt{a^2\,(1-\frac{x^2}{b^2})}}{r}\right), \tag{60}$$

$$Etan\,u \;= \frac{r\,sin\,u}{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}, \tag{61}$$

$$Esin\,u = \left(\frac{r\,sin\,u}{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}\right)Ecos\,u, \tag{62}$$

$$Ecos\,u \;= \left(\frac{\sqrt{b^2\,(1-\frac{y^2}{a^2})}}{r\,sin\,u}\right)Esin\,u, \tag{63}$$

Also, angle (u) is measured by the degree of rotation from the initial side to the terminal side of the ellipse. The direction of rotation, whether clockwise or counterclockwise, determines the sense of the angle, similar to the circular system. El-functions establish the relationship between Etangent, Ecotangent, Esecant, and Ecosecant in terms of Esine and Ecosine, providing elliptic counterparts to every circular function. This equivalence places the ellipse and circle on equal footing. As a closed curve, the ellipse is periodic, with values repeating every 2π radians, making elliptic functions a generalization of trigonometric functions.

Given the ellipse's definition, any point on the curve can be determined by the ray originating from the center point and passing through the El-identities of Esin u and Ecos u. From Fig.2, formulas relating r to Esin u and Ecos u can be used to calculate the length of the ray (r) for any ellipse, thus enabling the definition of points on the ellipse by assigning values of a and b.

$$\sin u = \frac{\cos u\,.Ein\,u}{Ecos\,u}, \tag{64}$$

$$\cos u = \frac{\sin u\,.Ecos\,u}{Esin\,u}, \tag{65}$$

$$\sin u = \cos u\,.\,Etan\,u, \tag{66}$$

$$\cos u = \sin u\,.\,Ecot\,u, \tag{67}$$

Stated differently, if a set of ellipses shares the same center point and major axis (b), they will have the same value of Ecos u. This can be understood from the fact that Ecos u is determined by the ratio of the distance from the center point to a point on the ellipse (r) and the length of the major axis (b), as described by the relevant formula. Since the center point and major axis remain constant for a given set of ellipses, the value of Ecos u will also remain constant for that set.

$$Etan\, u = a, \tag{68}$$

$$Ecot\, u = b, \tag{69}$$

Assuming that the El-functions of Etan u are equivalent to the length of the minor axis (a) of the ellipse, all ellipses with the same center point and major axis length (b) will have a constant value of Ecos u. Consequently, the vertical ray represents the locus of intersection points of rays drawn from the center to each point on the ellipse. The line segment defined by the intersection of this vertical ray with the ellipse divides the major axis (b) into two segments of different lengths, namely (b-Ecos u) and (Ecos u).

Discussions

This paper examines the E-trigonometry functions associated with a ray from the center point of an ellipse to a point on its perimeter, where (a) and (b) represent the major and minor axis of the ellipse. The aim is to present defining relations for the tangent, cotangent, secant, and cosecant in terms of Elliptic ratios (El-functions). The paper demonstrates that each of the six trig functions has a co-function of El-identities evaluated at the complementary angle and coordination. The paper showed that the similarity sense the periodicity of these new functions (Alías et al., 2016; P. Wang et al., 2018). The similarity between these functions and their periodicity is shown, where Esine, Ecosine, Esecantp, and Ecosecant have a period of 2π, while Etangent and Ecotangent have a period of π. The El-functions presented in this paper demonstrate periodic range and phases that are shared with circular functions, indicating similarity in arches and angles.

Using the Pythagorean formula and equations (1 & 2) to build a right angle triangle, the length of the ellipse radius (r) can be calculated in terms of (Esin u, Ecos u) for any point on the ellipse. This allows for precise measurement of the ray length from the ellipse's centre point given Esin u, Ecos u, a, b, and u. These equations can fit the ellipse over an infinite range of parameters (a, b, x, y, and u). This paper presents two cases and 69 formulas of these El-functions associated with those of trigonometric functions.

These El-functions can be used to compare a sum or difference of circular functions into a product of sines and cosines, which is useful for describing basic trig functions in terms of the tangent of half the angle (Glassmeyer et al., 2019; Hery & Ramamoorthi, 2012; J. Liu et al., 2012; Sterling, 2023). For advanced-level calculus, a particular kind of substitution in integrals is possible (Faulkner et al., 2020; Törner et al., 2014). Double-angle formulas for Esin and cosine are also available, allowing for deriving the other two from the Pythagorean formula. Although the El-functions have some future applications, they have narrow applications compared to circular functions and may be forgotten until needed. El trigonometry, however, can potentially solve real math and physics problems.

Acknowledgements

The author would like to express their profound gratitude to all those who have provided valuable assistance, moral support, and guidance during the development of this article, which has led to its successful publication. Special appreciation is extended to the authors' family members, including Angham S. Mehdi, Noor, Nada, Fatima, and Omneya, for their unwavering support and coordination.
In accordance with standard academic practices, the author declares that they have no conflicts of interest to disclose. Furthermore, they confirm that all figures, tables, and measurements included in the manuscript are their original work.

References

Al-ossmi, L. (2023). Elementary treatise on melted-like curves derived from center ellipse and circle. *Iraqi Journal For Computer Science and Mathematics*, *4*(1), 114–129. https://doi.org/10.52866/ijcsm.2023.01.01.009

Al-ossmi, L. H. M. (2022). Nada's curve towards a new curvature produced by the tangent of a circle and an ellipse: The nada's curve. *Iraqi Journal For Computer Science and Mathematics*, *4*(1), 1–9. https://doi.org/10.52866/ijcsm.2023.01.01.001

Alías, F., Socoró, J. C., & Sevillano, X. (2016). A review of physical and perceptual feature extraction techniques for speech, music and environmental sounds. *Applied Sciences*, *6*(5), 143. https://doi.org/10.3390/app6050143

Alquran, M., & Jarrah, A. (2019). Jacobi elliptic function solutions for a two-mode KdV equation. *Journal of King Saud University-Science*, *31*(4), 485–489. https://doi.org/10.1016/j.jksus.2017.06.010

Altman, R., & Kidron, I. (2016). Constructing knowledge about the trigonometric functions and their geometric meaning on the unit circle. *International Journal of Mathematical Education in Science and Technology*, *47*(7), 1048–1060. https://doi.org/10.1080/0020739X.2016.1189005

Bialy, M. (2022). Mather β-Function for ellipses and rigidity. *Entropy*, *24*(11), 1600. https://doi.org/10.3390/e24111600

Boyer, C. B., & Merzbach, U. C. (1968). A history of mathematics john wiley & sons. *Inc New York, London, Sydney*.

Byrd, P. F., & Friedman, M. D. (1971). *Handbook of elliptic integrals for engineers and scientists*. Berlin, Heidelberg.

Dimitrov, B. G. (2022). Elliptic integrals for calculation of the propagation time of a signal, emitted and received by satellites on one orbit. *AIP Conference Proceedings*, *2522*(1), 90004. https://doi.org/10.1063/5.0101261

Dragović, V., & Radnović, M. (2011). *Poncelet porisms and beyond: Integrable billiards, hyperelliptic Jacobians and pencils of quadrics*. Springer Science & Business Media. https://link.springer.com/book/10.1007/978-3-0348-0015-0

Faulkner, B., Johnson-Glauch, N., San Choi, D., & Herman, G. L. (2020). When am I ever going to use this? An investigation of the calculus content of core engineering courses. *Journal of Engineering Education*, *109*(3), 402–423. https://doi.org/10.1002/jee.20344

Glassmeyer, D., Brakoniecki, A., & M. Amador, J. (2019). Promoting uncertainty to support preservice teachers' reasoning about the tangent relationship. *International Journal of Mathematical Education in Science and Technology*, *50*(4), 527–556. https://doi.org/10.1080/0020739X.2018.1527405

Hazewinkel, M. (2001). Jacobi elliptic functions. *Encyclopedia of Mathematics, Springer*.

Hery, C., & Ramamoorthi, R. (2012). Importance sampling of reflection from hair fibers. *Journal of Computer Graphics Techniques (JCGT)*, *1*(1), 1–17. https://graphics.pixar.com/library/ImportanceSamplingHair/paper.pdf

Kim, D., & Kim, Y. (2021). Causal image retrieval with ellipse-based object proposal. *Journal of Visual Communication and Image Representation*, *75*(1), 102855.

Li, Y., Wu, L., & Li, L. (2021). Research on a new multiple criteria decision making method based on the ellipse membership function. *Journal of Intelligent & Fuzzy Systems*, *41*(1), 1357–1371.

Liu, J., Shao, Y., & Lim, T. C. (2012). Vibration analysis of ball bearings with a localized defect applying piecewise response function. *Mechanism and Machine Theory*, *56*, 156–169. https://doi.org/10.1016/j.mechmachtheory.2012.05.008

Liu, Q., & Gao, Y. (2021). The optimal approximation of the eccentricity of the ellipse with constrained arc length. *Journal of Applied Mathematics and Computing*, *68*(12), 653–669.

Liu, S., Fu, Z., Liu, S., & Zhao, Q. (2001). Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. *Physics Letters A*, *289*(1–2), 69–74. https://doi.org/10.1016/S0375-9601(01)00580-1

Liu, S. S., Fu, Z. T., & D., L. S. (2001). Expansion method about the jacobi elliptic function and its applications to nonlinearwave E2 quations. *Acta Physici Sinica (in Chinese)*, *50*(11), 2068–2073. https://doi.org/10.7498/aps.50.2068

Reznik, D., Garcia, R., & Koiller, J. (2021). Fifty new invariants of n-periodics in the elliptic billiard. *Arnold Mathematical Journal*, *7(1)*, 341–355. https://doi.org/10.1007/s40598-021-00174-y

Salas, A. H., Abu Hammad, M., Alotaibi, B. M., El-Sherif, L. S., & El-Tantawy, S. A. (2022). Closed-form solutions to a forced damped rotational pendulum oscillator. *Mathematics*, *10*(21), 4000. https://doi.org/10.3390/math10214000

Schaum, A. (2009). Advanced hyperspectral detection based on elliptically contoured distribution models and operator feedback. *2009 IEEE Applied Imagery Pattern Recognition Workshop (AIPR 2009)*, 1–5. https://doi.org/10.1109/AIPR.2009.5466308

Shen, M., Wang, J., Joseph, A., Peng, F. Z., Tolbert, L. M., & Adams, D. J. (2006). Constant boost control of the z-source inverter to minimize current ripple and voltage stress. *IEEE Transactions on Industry Applications*, *42*(3), 770–778.

https://doi.org/10.1109/TIA.2006.872927

Siyepu, S. W. (2015). Analysis of errors in derivatives of trigonometric functions. *International Journal of Stem Education*, *2*, 1–16. https://doi.org/10.1186/s40594-015-0029-5

Smart, J. R., & Schwandt, E. A. (2012). *Elliptic modular functions: An introduction* (Vol. 203). Springer Science & Business Media. https://doi.org/10.1007/978-3-642-65663-7

Sterling, M. J. (2023). *Trigonometry for dummies*. John Wiley & Sons. https://cds.cern.ch/record/2018988

Törner, G., Potari, D., & Zachariades, T. (2014). Calculus in European classrooms: curriculum and teaching in different educational and cultural contexts. *ZDM*, *46*, 549–560. https://doi.org/10.1007/s11858-014-0612-0

Turner, H. R. (2010). *Science in medieval Islam: An illustrated introduction*. University of Texas Press.

Wang, B., & Jin, G. (2022). Solving a class of nonlinear evolution equations using jacobi elliptic functions. *Journal of Physics: Conference Series*, *2381*(1), 12038. https://doi.org/10.1088/1742-6596/2381/1/012038

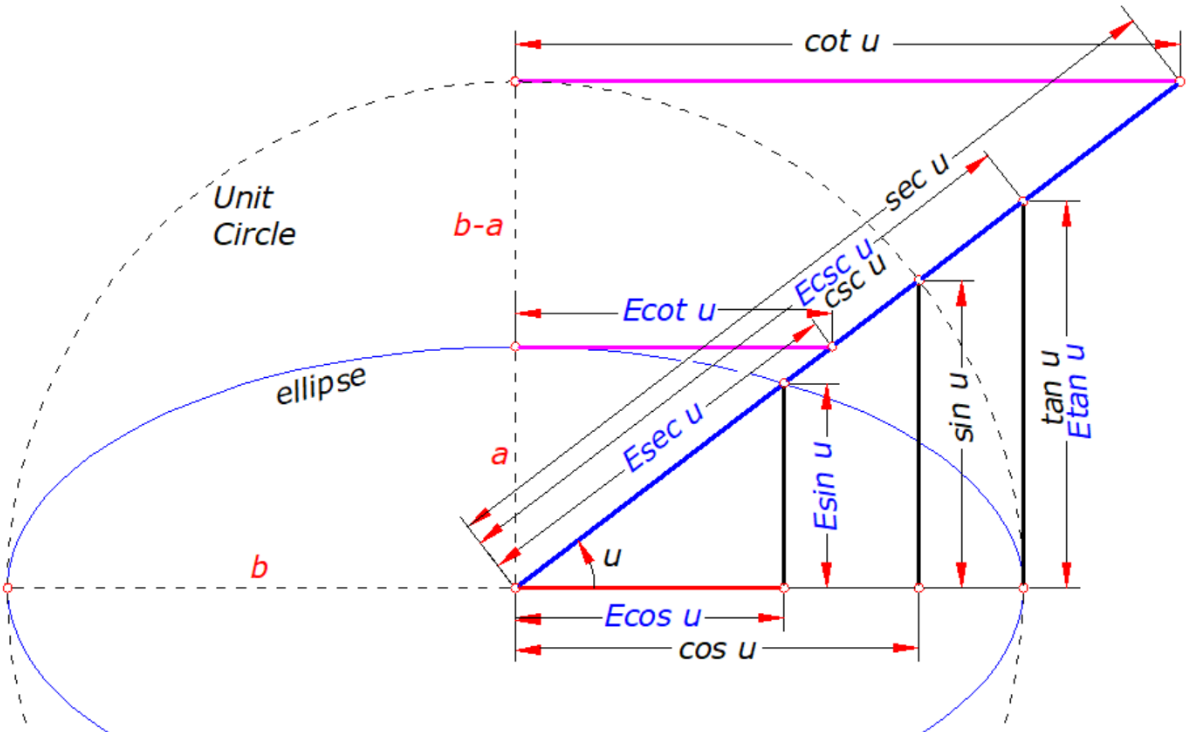
Wang, P., Fu, Y., Zhang, J., Li, X., & Lin, D. (2018). Learning urban community structures: A collective embedding perspective with periodic spatial-temporal mobility graphs. *ACM Transactions on Intelligent Systems and Technology (TIST)*, *9*(6), 1–28. https://doi.org/10.1145/3209686

Wang, W., Stuijk, S., & De Haan, G. (2015). A novel algorithm for remote photoplethysmography: Spatial subspace rotation. *IEEE Transactions on Biomedical Engineering*, *63*(9), 1974–1984. https://doi.org/10.1109/TBME.2015.2508602

Yang, C., Wang, J., & Liu, X. (2021). An efficient nonlocal algorithm for ellipse detection. *Signal Processing*, *187*(1), 108127.

Attachment 1:

Appendix A:



Picture 1. By fixing the angle (u) and the major axis of the ellipse is the radius of the Unit Circle, (b).

Attachment 2:

Table 1. E-identities when angle ($u = 0$)

a	*Etan u*	*Esin u*	*Ecos u*	*Ecsc u*	*Esec u*	*Ecot u*
a	0.00000	0.00000	b	∞	0.00000	∞

Table 2. E-identities when angle ($u = 5$)

a	*Etan u*	*Esin u*	*Ecos u*	*Ecsc u*	*Esec u*	*Ecot u*
0	0.00000	0.00000	b	0.00000	0.00000	0.00000
0.1 *b*		0.068881b	0.724939b	1.057188b		1.052447b
0.2 *b*		0.085824b	0.903248b	2.114375b		2.104895b
0.3 *b*		0.090582b	0.953327b	3.171563b		3.157342b
0.4 *b*		0.092444b	0.972927b	4.228750b		4.209790b
0.5 *b*	0.095017b (*tan u*)	0.093346b	0.982418b	5.285938b	1.004504b (*csc u*)	5.262237b
0.6 *b*		0.093847b	0.987692b	6.343126b		6.314685b
0.7 *b*		0.094153b	0.990913b	7.400313b		7.367132b
0.8 *b*		0.094353b	0.993021b	8.457501b		8.419580b
0.9 *b*		0.094491b	0.994473b	9.514689b		9.472027b
b		*sin u*	*cos u*	*sec u*		*cot u*

Where (a) and (b) are the major and minor axis of the ellipse.

Table 3. E-identities when angle ($u = 10$)

a	*Etan u*	*Esin u*	*Ecos u*	*Ecsc u*	*Esec u*	*Ecot u*
0	0.00000	0.00000	b	0.00000	0.00000	0.00000
0.1 *b*		0.086303b	0.505156b	0.593811b		0.58533b
0.2 *b*		0.129902b	0.760354b	1.187621b		1.17066b
0.3 *b*		0.148458b	0.868972b	1.781432b		1.75599b
0.4 *b*		0.157113b	0.919631b	2.375243b		2.34132b
0.5 *b*	0.170844b (*tan u*)	0.161667b	0.946285b	2.969054b	1.014489b (*csc u*)	2.92665b
0.6 *b*		0.164313b	0.961771b	3.562864b		3.51198b
0.7 *b*		0.165972b	0.971485b	4.156675b		4.09731b
0.8 *b*		0.167076b	0.977949b	4.750486b		4.68264b
0.9 *b*		0.167846b	0.982456b	5.344297b		5.26797b
b		*sin u*	*cos u*	*sec u*		*cot u*

Where (a) and (b) are the major and minor axis of the ellipse.