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DERIVING THE EXACT FORMULA FOR PERIMETER OF AN ELLIPSE USING COORDINATE TRANSFORMATION

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Abstract:
The ellipse can be transformed into a circle by dilating the coordinates of the ellipse relative to the *x*-axis and *y*-axis. Therefore, this study aimed to derive the formula for the equation of the perimeter of an ellipse by using the transformation of an ellipse to a circle. This transformation was arranged so that the perimeter of the ellipse was equal to the perimeter of the circle. The type of research was in the review of books, articles, and relevant research reports. The results showed that the ellipse can be transformed into a circle while maintaining its perimeter. So, the perimeter of the ellipse was the same as the perimeter of the circle.

Keywords: Coordinate transformation, approximation, formula, perimeter of an ellipse, Integral Elliptic.

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INTRODUCTION

Until now there is no definite formula for determining the perimeter of an ellipse, some mathematicians have only discovered and perfected how to calculate the perimeter of an ellipse (k(E)) using the approximation method. It is difficult to determine the exact formula for the perimeter of an ellipse, because of the difficulty in integrating $K(E) = 4\int_0^{\pi/2} \sqrt{(a\cos t)^2 + (b\sin t)^2}\,dt$, where $a > b > 0$. This integral value can only be approximated by using the approximation method. This function $\sqrt{(a\cos t)^2 + (b\sin t)^2}$ has no antiderivative which can be written in the form of basic calculus functions (B. Thomas, 2018; J. Purcell & Varberg, 2016a).

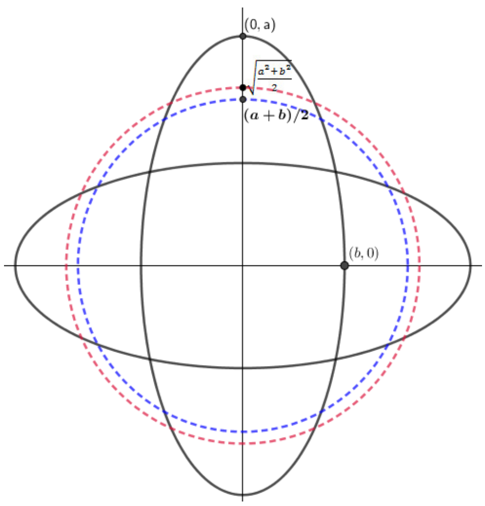
Researchers have so far found that the formula for the perimeter of an ellipse is an infinite series in the form of a hyper-geometric function and tends to have a high degree of error. The formula for the perimeter of an ellipse is more commonly known as the elliptic integral. Some of the hyper-geometric functions that have been found by experts to calculate the perimeter of an ellipse in more detail can be seen in the article written by (Abbott, 2009; Adlaj, 2012; Almkvist & Berndt, 1988; Alzer & Qiu, 2004; B. Villarino, 2008; Barnard, Pearce, & Schovanec, 2001). Ramanujan was one of the mathematicians who succeeded in determining the formula for the integral $K(E)$ with the least error rate (Almkvist & Berndt, 1988).

As we know, a circle is an ellipse that has the same major and minor axes. The perimeter of the ellipse, $K(E)$ with $a = b > 0$, can be found easily using definite integrals. Therefore, one way to find the integral for with is to equate the lengths of the major and minor axes to the radius of the circle (c). This method can only be done if we transform the ellipse into a circle of radius c while maintaining the distance between the coordinates.

In previous research, Mazer (2010) has provided an example of a matrix formula for transforming a circle into an ellipse, and vice versa. This transformation shows that the ellipse coordinates can be transformed into circular coordinates relative to the x-axis and y-axis (Lockhart, 2012; Mazer, 2010). However, this transformation only applies to circles of radius 1 and the discussion does not continue about the relationship between the circumference of an ellipse and a circle.

In transforming an ellipse into a circle of radius $c > 0$, we need a bound of c. The bound of c can guarantee that the perimeter of the ellipse and circle is the same. We can get the radius of this circle from the perimeter of the ellipse, i.e. $2\pi(a+b)/2 \leq K(E) \leq 2\pi\sqrt{(a^2+b^2)/2}$, which has been by previous research (E. Pfiefer, 1988; Gusić, 2015; GJO Jameson, 2015). If these boundaries of K(E) are circles, then we get the bounds of the radius of the circle (c) i.e., $(a+b)/2 \leq c \leq \sqrt{(a^2+b^2)/2}a$ with $\geq b > 0$.

E. Pfiefer (1988) has also proved that $K(E)$is the mean of the perimeter of a vertical ellipse $(x^2/b^2 + y^2/a^2 = 1)$ and a horizontal ellipse $(x^2/a^2 + y^2/b^2 = 1)$. If P_v is the perimeter of the vertical ellipse and P_h is the perimeter of the horizontal ellipse then $2\pi(a+b)/2 \leq (P_v + P_h)/2 = K(E) \leq 2\pi\sqrt{(a^2+b^2)/2}$ (E. Pfiefer, 1988; Gusić, 2015; Jameson, 2015). Thus, the average of the perimeters of the vertical and horizontal ellipses is at the perimeter of the circle of radius c, $(a+b)/2 \leq c \leq \sqrt{(a^2+b^2)/2}$. Picture 1 shows a vertical and horizontal ellipse of the same size, a circle with a radius $(a+b)/2$ (blue dashed line) and a radius circle $\sqrt{(a^2+b^2)/2}$ (red dashed line).



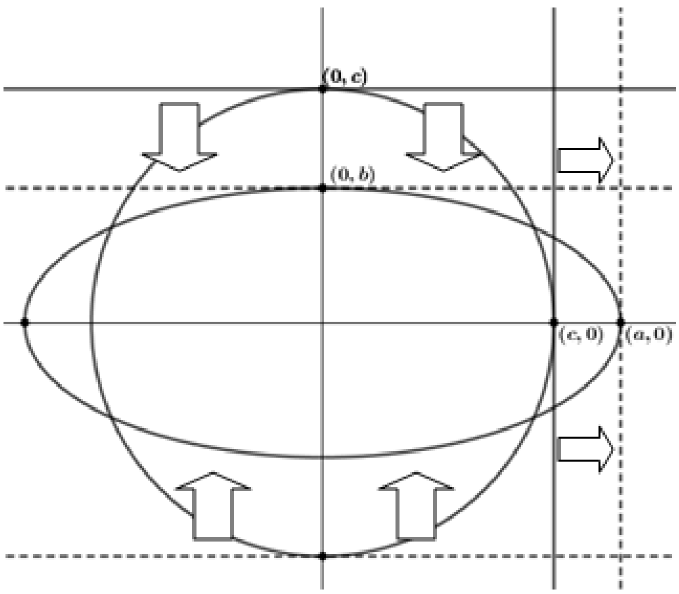
Picture 1. The perimeter of an ellipse

Based on the dilatation of the ellipse coordinates relative to the *x*-axis and *y*-axis, the ellipse can be transformed into a circle (Archimedes, 2010; Barnard et al., 2001; Lockhart, 2012; Mazer, 2010; Parker & Pryor, 1944; Rohman & Jupri, 2019). One of them is to transform the ellipse into a circle with radius *c*. If the radius of this circle is $(a+b)/2 \leq c \leq \sqrt{(a^2+b^2)/2}$, then we can confirm that the perimeter of the ellipse is $2\pi(a+b)/2 \leq K(E) \leq 2\pi\sqrt{(a^2+b^2)/2}$ or we find the correct value of *c* to show that $K(E) = 2\pi c$ is the perimeter of the ellipse.

For example, transform the coordinates of a circle (x, y) with radius a into ellipse coordinates $(y_e = (b/a)y, x_e)$, where b and a are the semi-major and semi-minor lengths. Dilation in the ordinate or abscissa only causes the perimeter of the ellipse and circle to be different. Therefore, we need the composition of the transformation of the abscissa and ordinate coordinates such that the distance between the coordinates is constant and results in a fixed perimeter of the circle and ellipse.

Transforming an ellipse into a circle by maintaining its perimeter we can imagine the perimeter of the ellipse as a thread. We can change the thread that forms the perimeter of this circle into any closed-plane geometry shape, of course, the perimeter is still the same. One example is that we can change the thread around the circle to an ellipse, so we can determine if the perimeter of the ellipse is the same as the circle. Vice versa, we can change the thread that forms the perimeter of the ellipse into a circle.

Another example, suppose we have an elastic object in the form of a circle, then we press inward from both sides vertically, there will be a horizontal stretch. In simple terms, this can be illustrated in picture 2 below. When there is an emphasis on the x-axis or vertically, there will be a horizontal stretch that causes the circle to become an ellipse. Moreover, we can guarantee that the perimeter is constant.



Picture 2. Circle to Ellipse Transformation

Changing a circle to an ellipse or vice versa by maintaining its perimeter requires a composition of coordinate transformations such that the perimeter is always constant. In other words, we must keep the distance between the coordinates of the ellipse constant when it is transformed into a circle or vice versa. The change of an ellipse into a circle with a fixed perimeter must satisfy $2\pi(a+b)/2 \leq K(E) \leq 2\pi\sqrt{(a^2+b^2)/2}$. Therefore, this research was conducted to: 1) determine the types of coordinate transformations from ellipse to circle or vice versa to maintain the distance between the coordinates, 2) by maintaining the distance between the coordinates, it can be ascertained that the perimeter of the polygon is always constant, and 3) When we can ensure that the perimeter of the inscribed polygon is always constant, then we can use the perimeter of the ellipse by using the perimeter approximation of the circle.

RESEARCH METHODS

This research is based on the review of books, articles, and the relevant research reports that have been done by (Archimedes, 2010; E. Pfiefer, 1988; Gusić, 2015; Hilbert & Cohn-Vossen, 2021; J. Purcell & Varberg, 2016b; Jameson, 2014; Lockhart, 2012; Mazer, 2010; Rohman & Jupri, 2019). Through the Minkowski Sums rule, i.e., $2\pi(a+b)/2 \leq K(E) \leq 2\pi\sqrt{(a^2+b^2)/2}$, and coordinate transformation, the researcher tries to transform ellipse coordinates to circle while maintaining the perimeter of the ellipse.

The stages carried out in this research are:

1. Determine the transformation of the coordinates of an ellipse to a circle

   At this stage, the researcher determines the transformation formula in the form of abscissa (x) and ordinate (y) dilation, which are relative to the x-axis and y-axis, respectively, so that the results of the coordinate dilatation when substituted into the ellipse equation produce a circle equation.

2. Determine the radius of the circle so that the polygons in the ellipse and the circle are the same lengths

   At this stage, the researcher determines the radius of the circle (c) which is taken from $K(E)$ which is at $(a+b)/2 \leq c \leq \sqrt{(a^2+b^2)/2}$. The researcher first tested the value of the lower and upper bounds, namely $(a+b)/2$ and $\sqrt{(a^2+b^2)/2}$ which causes the length of the polygon in the ellipse to be the same as the length of the polygon in the circle when the ellipse is transformed into a circle of radius c. If none of the lower and upper bounds meet (maintain the length of the polygon), then look for another value of c that satisfies.

3. Find the formula for the perimeter of an ellipse

   At this stage, the researcher has produced a value of c that meets stage 2, then the researcher derives the formula for the perimeter of the ellipse by using an approach to the perimeter of a polygon in a circle.

RESULTS AND DISCUSSION

Transforming Circle Coordinates to Ellipse

Suppose a horizontal ellipse is an ellipse whose long axis coincides with the *x*-axis and has coordinates (x_h, y_h) with the following equation.

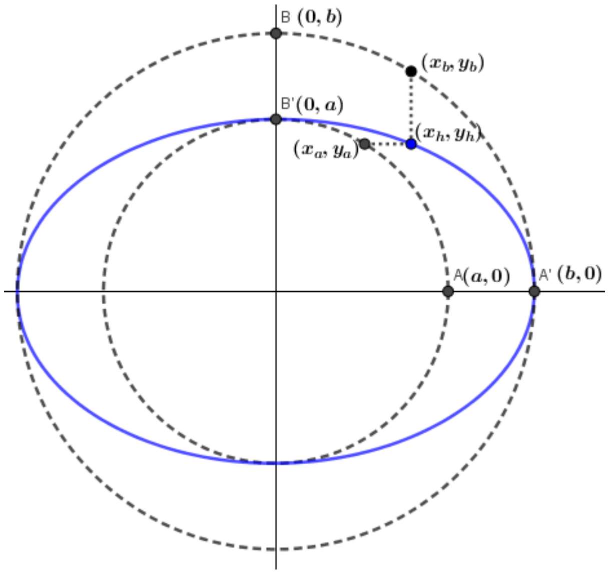
$$\frac{x_h^2}{b^2} + \frac{y_h^2}{a^2} = 1$$

or

$$y_h = \frac{a}{b}\sqrt{b^2 - x_h^2}$$

where *2b* and *2a* are the major and minor axes (J. Purcell & Varberg, 2016b).

Based on the equation y_h, the ordinate of the circle $y_b = \sqrt{b^2 - x_h^2}$ is dilated by a/b relative to the *x*-axis, while $x_h = x_b$ (A. Brannan, F. Esplen, & J. Gray, 2012; Rohman & Jupri, 2019). As seen in picture 3, the coordinates of a circle with radius b $(x_b, y_b)$are transformed into horizontal ellipse coordinates (x_h, y_h) (blue line).



Picture 3. Horizontal ellipse and circle

As seen in Picture 3, the horizontal ellipse can also be viewed as the result of the transformation of the coordinates of a circle (x_a, y_a), with radius *a,* be the ellipse coordinates (x_h, y_h) with $x_h = \frac{b}{a}x_a$ and $y_a = y_h$. The general equation for the same horizontal ellipse will be generated, namely.

$$x_h = \frac{b}{a}x_a = \frac{b}{a}\sqrt{a^2 - y^2} = \frac{b}{a}\sqrt{a^2 - y_h^2}$$

or can be written

$$\frac{x_h^2}{b^2} + \frac{y_h^2}{a^2} = 1$$

Horizontal Ellipse Transform to Vertical Ellipse

There are two ways to transform the coordinates of a horizontal ellipse $(x_h^2/b^2 + y_h^2/a^2 = 1)$ to a vertical ellipse $(x_v^2/a^2 + y_v^2/b^2 = 1)$ with b>a>0. The methods are:

a. Elliptical Coordinate Dilation

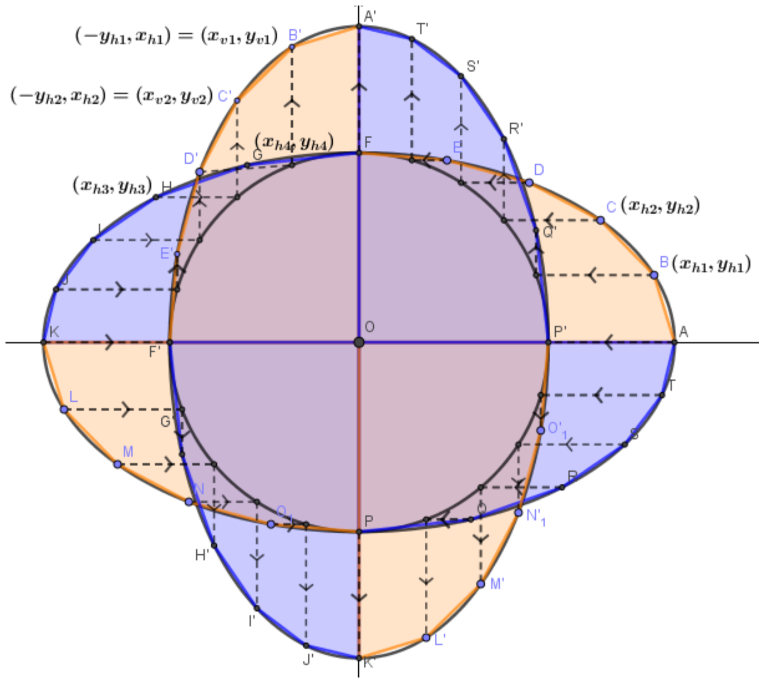
The coordinates of the horizontal ellipse (x_h, y_h) are transformed into the coordinates of a circle with a radius a (x_a, y_a) then transformed into vertical ellipse coordinates (x_v, y_v). As seen in picture 4, the point is $E(x_h, y_h)$ translated $G(x_a, y_a)$ into and finally becomes $J(x_v, y_v)$.

The composition rule for transforming horizontal ellipse coordinates into vertical ellipse coordinates, where b>a>0, can be written as

$$(x_h, y_h) \xrightarrow{x_a = \frac{a}{b} x_h} \left(x_a = \frac{a}{b} x_h, y_a\right) \xrightarrow{y_v = \frac{b}{a} y_h} \left(x_a, y_v = \frac{b}{a} y_h\right) = (x_v, y_v) \quad (1)$$

or if written in a matrix is

$$\begin{bmatrix} x_v \\ y_v \end{bmatrix} = \begin{bmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} \frac{a}{b} x_h \\ \frac{b}{a} y_h \end{bmatrix} = \begin{bmatrix} x_a \\ \frac{b}{a} y_a \end{bmatrix} \quad (2)$$



Picture 4. Horizontal Ellipse Transformation to Vertical Ellipse while Maintaining Its Size

We can get the transformation of the horizontal ellipse equation into a vertical ellipse by entering equation (2) into the horizontal ellipse equation $x_h^2/b^2 + y_h^2/a^2 = 1$, we get

$$\frac{\left(\frac{b^2}{a^2}\right)x_v^2}{b^2} + \frac{\left(\frac{a^2}{b^2}\right)y_v^2}{a^2} = \frac{x_v^2}{a^2} + \frac{y_v^2}{b^2} = 1, b > a > 0$$

b. Rotation $(R\left(0, \frac{1}{2}\pi\right))$

The horizontal ellipse coordinates are rotated counterclockwise $(R\left(0, \frac{1}{2}\pi\right))$ as shown in picture 4. Transforming horizontal ellipse coordinates into vertical ellipse coordinates can be written

$$(x_h, y_h) \xrightarrow{R\left(0, \frac{1}{2}\pi\right)} (-y_h, x_h) = (x_v, y_v) \tag{3}$$

or it can be written in a matrix is

$$\begin{bmatrix} x_v \\ y_v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} -y_h \\ x_h \end{bmatrix} \tag{4}$$

so that we get $x_v = -y_h$ and $y_v = x_h$.

We can get the transformation of the horizontal ellipse equation into a vertical ellipse by entering equation (4) into the horizontal ellipse equation $(x_h^2/b^2 + y_h^2/a^2 = 1)$, we get

$$\frac{x_h^2}{b^2} + \frac{y_h^2}{a^2} = 1$$

because $x_v = -y_h$ and $y_v = x_h$, then

$$\frac{y_v^2}{b^2} + \frac{x_v^2}{a^2} = 1, b > a > 0$$

Since equations (2) and (4) produce the same coordinates, we can write the transformation composition of a vertical ellipse to a horizontal ellipse as follows

$$T_e = \begin{bmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

In addition, from equations (2) and (4) we get that

$$\begin{bmatrix} x_v \\ y_v \end{bmatrix} = \begin{bmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} \frac{a}{b}x_h \\ \frac{b}{a}y_h \end{bmatrix} = \begin{bmatrix} -y_h \\ x_h \end{bmatrix} \tag{6}$$

By using equation (6), we can guarantee that the horizontal ellipse $(x_h^2/b^2 + y_h^2/a^2 = 1)$ which is transformed by T_s becomes a vertical ellipse $(x_v^2/a^2 + y_v^2/b^2 = 1)$ and has the same size (isometric).

Look at Picture 4, which $\overline{B'C'}$ is the result of dilation $\overline{GH}$ by equation (2) and also the result of rotation $\overline{BC}$ by equation (4). We can guarantee that the segment length $\overline{BC}$ is equal to the segment length $\overline{B'C'}$ ($m\overline{BC} = m\overline{B'C'}$) by using equation (6). If $m\overline{BC}$ and $m\overline{B'C'}$ are respectively the lengths of one of the sides of the polygon in the horizontal and vertical ellipse, i.e. Δp_h and Δp_v , we get

$$\Delta p_h = \sqrt{\Delta x_h^2 + \Delta y_h^2}$$

Because $x_v = -y_h$ and $y_v = x_h$, then $\Delta x_v = \Delta y_h$ and $\Delta y_v = \Delta x_h$, so that we get

$$\Delta p_h = \sqrt{\Delta x_h^2 + \Delta y_h^2} = \sqrt{\Delta y_v^2 + \Delta x_v^2} = \Delta p_v \tag{7}$$

So that the perimeter of the horizontal and vertical ellipse can be found using the polygon perimeter approximation, namely

$$\sum_i^n \Delta p_{h_i} = \sum_i^n \Delta p_{v_i}$$

with $\Delta p_{h_1} = \Delta p_{h_2} = \Delta p_{h_3} = \cdots = \Delta p_{h_n}$ and $\Delta p_{v_1} = \Delta p_{v_2} = \Delta p_{v_3} = \cdots = \Delta p_{v_n}$ for $i = 1$ and $n \to \infty$ which gives an approximation that the perimeter of the vertical and horizontal ellipses are the same.

Transformation of Vertical Ellipse to Circle with Radius c

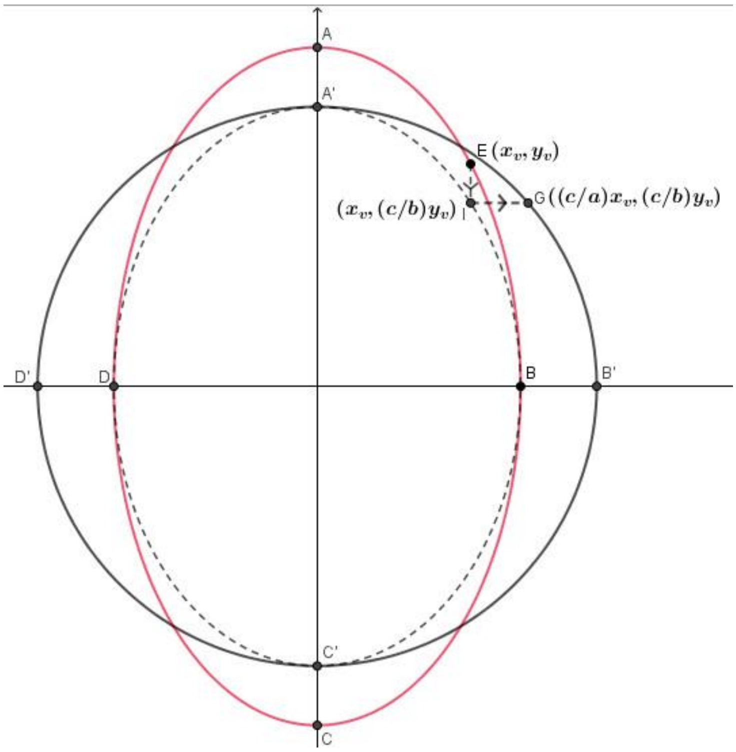
We can transform a vertical ellipse into a circle by using its transformation rules (Mazer, 2010). Suppose that the vertical ellipse coordinates (x_v, y_v) are transformed into circular coordinates (x_c, y_c) with the transformation rule, namely

$$(x_v, y_v) \xrightarrow{x_c = c/a)x_v \text{ and } y_c = (c/b)y_v} (x_c, y_c)$$

or it can be in the form of a matrix that is

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} \frac{c}{a} & 0 \\ 0 & \frac{c}{b} \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} \frac{c}{a} x_v \\ \frac{c}{b} y_v \end{bmatrix} \tag{8}$$

where c is the radius of the circle, b is the semi-major axis of the ellipse, and a is the semi-minor axis of the ellipse.



Picture 5. Elliptical to Circle Transform

We can derive the equation of a circle by substituting $x_v = \frac{a}{c}x_c$ and $y_v = \frac{b}{c}y_c$ into the equation $(x_v/a)^2 + (y_v/b)^2 = 1$ and we get

$$\left(\frac{a}{c}x_c\right)^2 + \left(\frac{b}{c}y_c\right)^2 = 1$$

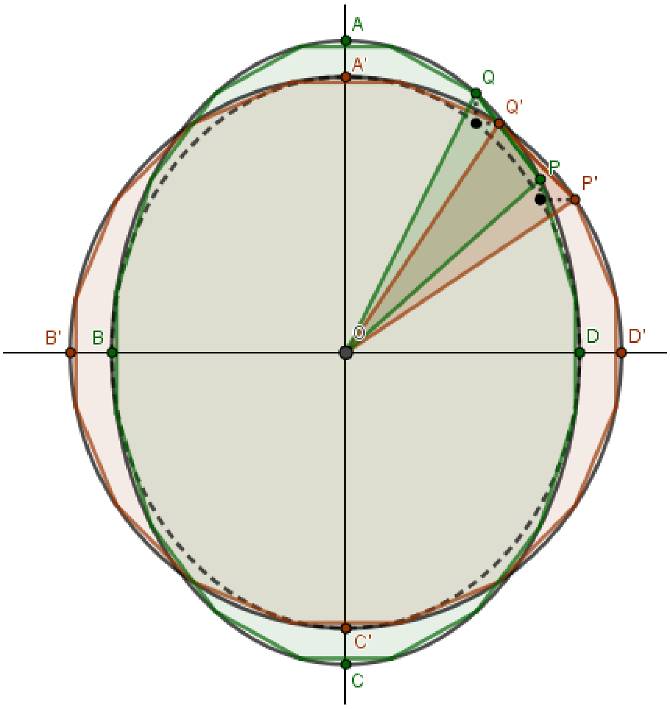
by multiplying both sides by c^2, we get the equation of the circle

$$x_c{}^2 + y_c{}^2 = c^2$$

Transform polygons in ellipses into circle polygons

As we already know that the way to find the length of the curve in continuous function that defined in an interval is to divide the curve into several parts and we approximate each part using straight line segments. The sum of each segment is the approximate length of the curve. An example of calculating the perimeter of a circle is to use the approximation of regular polygons (n-gons) in a circle as n approaches infinity ($n \to \infty$). Likewise, with an ellipse, we can calculate the perimeter of an ellipse by using the polygon approximation in an ellipse where each side is the same length.

It is known that an ellipse can be transformed into a circle of a certain radius. One way to find the perimeter of an ellipse is to transform it into a circle while maintaining the length of the polygon. So, we get that the polygons (n-gons) in the ellipse and circle have the same n and the same length of each side. Furthermore, we can calculate the perimeter of the ellipse by using the approximation of the perimeter of the polygon in the circle.



Picture 6. Elliptical Polygon Transformation to Circle Polygon

If the lengths of each side of the polygon in the circle and ellipse are ΔP_c and ΔP_v with $\Delta P_c = \Delta P_v$, then the perimeter of the ellipse vertical is (you can use an approximation of the perimeter of the circle, i.e.,)

$$P_v = \sum_{i}^{n} \Delta P_{c_i} = P_c$$

for $i = 1$ and $n \to \infty$. As shown in Picture 6, the ellipse, which has major axis *AC* and minor axis *BD,* is transformed into a circle with radius *OA'*. An ellipse that has an inscribed polygon which each side is the same length transformed into an \ inscribed polygon of a circle where each side is still the same length, one of which is m$\overline{PQ} = m\overline{P'Q'}$.

The transformation of the ellipse coordinates into circular coordinates by maintaining the length of the polygon on each side, so that $P_v = P_c$ must satisfy $2\pi(a+b)/2 \le P_v \le 2\pi\sqrt{(a^2+b^2)/2}$ (E. Pfiefer, 1988). Based on these bounds, the ellipse can be transformed into a circle with radius $(a+b)/2 \le c\ (radius) \le \sqrt{(a^2+b^2)/2}$.

a. Transforming Elliptical Polygons to Circle Polygons of Radius $\sqrt{(a^2+b^2)/2}$

Suppose a circle, whose radius is *c*, has coordinates (x_c, y_c) and the length of one side of the regular polygon in the circle is $\Delta P_c = \sqrt{\Delta x_c^2 + \Delta y_c^2}$. Coordinates (x_c, y_c) can be obtained from the transformation of ellipse coordinates, that is

$(x_c, y_c) = (x_c = (c/a)x_v, y_c = (c/b)y_v)$, so that we can write the length of one side of a regular polygon in a circle is $\Delta P_c = \sqrt{(c/a)^2\,\Delta x_v^2 + (c/b)^2 \Delta y_v^2}$. Next, we choose $c = \sqrt{(a^2 + b^2)/2}$ and get

$$\Delta P_c = \sqrt{\left(\frac{a^2+b^2}{2a^2}\right)\Delta x_v^2 + \left(\frac{a^2+b^2}{2b^2}\right)\Delta y_v^2}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\left(1+\frac{b^2}{a^2}\right)\Delta x_v^2 + \left(\frac{a^2}{b^2}+1\right)\Delta y_v^2\right]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\frac{b^2}{a^2}\Delta x_v^2 + \frac{a^2}{b^2}\Delta y_v^2 + \Delta x_v^2 + \Delta y_v^2\right]}$$

because based on equation (2) i.e. $\begin{bmatrix} x_v \\ y_v \end{bmatrix} = \begin{bmatrix} y_h \\ x_h \end{bmatrix} = \begin{bmatrix} (a/b)x_h \\ (b/a)y_h \end{bmatrix}$ or $x_v = (a/b)x_h$ and $y_v = (b/a)y_h$, we get $\Delta x_v = (a/b)\Delta x_h$ and $\Delta y_v = (b/a)\Delta y_h$, thus we substitute Δx_v and Δy_v into the equation ΔP_c, we get

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\frac{b^2}{a^2}\left(\frac{a}{b}\right)^2 \Delta x_h^2 + \frac{a^2}{b^2}\left(\frac{b}{a}\right)^2 \Delta y_h^2 + \Delta x_v^2 + \Delta y_v^2\right]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}[\Delta x_h^2 + \Delta y_h^2 + \Delta x_v^2 + \Delta y_v^2]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}[\Delta P_h^2 + \Delta P_v^2]}$$

and based on equation (7) that the length of the polygon on the horizontal and vertical ellipse is the same, that is $\Delta P_h = \Delta P_v$, then

$$\Delta P_c = \sqrt{\tfrac{1}{2}[2\Delta P_v^2]} = \Delta P_v \text{ or}$$

$$\Delta P_c = \sqrt{\tfrac{1}{2}[2\Delta P_h^2]} = \Delta P_h \qquad (9)$$

The selection $c = \sqrt{(a^2+b^2)/2}$ results in the perimeter of the polygon in the ellipse and the circle having the same perimeter. This shows that the perimeter of an ellipse can be approximated using a polygon of the perimeter of a circle with radius $c = \sqrt{(a^2+b^2)/2}$. As we already know that determining the perimeter of a circle using the approximation of the perimeter of a regular polygon in a circle whose number of sides approaches infinity (J. Purcell & Varberg, 2016).

$$\sum_i^n \Delta P_{c_i} = \sum_i^n \Delta P_{v_i}$$

with

$\Delta p_{c_1} = \Delta p_{c_2} = \Delta p_{c_3} = \cdots = \Delta p_{c_n}$,
$\Delta p_{v_1} = \Delta p_{v_2} = \Delta p_{v_3} = \cdots \Delta p_{v_n}$, and
$\Delta p_{c_i} = \Delta p_{v_i}$ for $i = 1$ and $n \to \infty$

So that the perimeter of a circle with radius c (P_c) equals the perimeter of a vertical ellipse (P_v) as follows:

$$P_c = P_v = 2\pi c = 2\pi\sqrt{\frac{a^2+b^2}{2}} = \pi\sqrt{2(a^2+b^2)} \qquad (10)$$

where b and a are semi-major and semi-minor axes, respectively.

b. Transform an ellipse into a circle with radius $c = (a + b/2)$

If we take$c = (a+b)/2$, we get

$$\Delta P_c = \sqrt{\left(\frac{c}{a}\right)^2 \Delta x_v^2 + \left(\frac{c}{b}\right)^2 \Delta y_v^2}$$

$$\Delta P_c = \sqrt{\left(\frac{a+b}{2a}\right)^2 \Delta x_v^2 + \left(\frac{a+b}{2b}\right)^2 \Delta y_v^2}$$

$$\Delta P_c = \sqrt{\frac{a^2+b^2+2ab}{4a^2}\Delta x_v^2 + \left(\frac{a^2+b^2+2ab}{4b^2}\right)\Delta y_v^2}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\frac{a^2+b^2+2ab}{2a^2}\Delta x_v^2 + \left(\frac{a^2+b^2+2ab}{2b^2}\right)\Delta y_v^2\right]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\left(\frac{1}{2}+\frac{b^2}{2a^2}+\frac{b}{a}\right)\Delta x_v^2 + \left(\frac{a^2}{2b^2}+\frac{1}{2}+\frac{a}{b}\right)\Delta y_v^2\right]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\frac{b^2}{2a^2}\Delta x_v^2 + \frac{a^2}{2b^2}\Delta y_v^2 + \frac{1}{2}\Delta x_v^2 + \frac{1}{2}\Delta y_v^2 + \frac{b}{a}\Delta x_v^2 + \frac{a}{b}\Delta y_v^2\right]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\frac{1}{2}\Delta P_h^2 + \frac{1}{2}\Delta P_v^2 + \frac{b}{a}\Delta x_v^2 + \frac{a}{b}\Delta y_v^2\right]}$$

based on equation (7) that is $\Delta P_h = \Delta P_v$ and based on equation (2) that is $\Delta x_v = (a/b)\Delta x_h$ and $\Delta y_v = (b/a)\Delta y_h$ then

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\Delta P_v^2 + \frac{b}{a}\Delta x_v^2 + \frac{a}{b}\Delta y_v^2\right]} \le \sqrt{\frac{1}{2}\left[\Delta P_v^2 + \left(\frac{b}{a}\right)^2 \Delta x_v^2 + \left(\frac{a}{b}\right)^2 \Delta y_v^2\right]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\Delta P_v^2 + \frac{b}{a}\Delta x_v^2 + \frac{a}{b}\Delta y_v^2\right]} \le \sqrt{\frac{1}{2}[\Delta P_v^2 + \Delta x_h^2 + \Delta y_h^2]}$$

$$\Delta P_c = \sqrt{\frac{1}{2}\left[\Delta P_v^2 + \frac{b}{a}\Delta x_v^2 + \frac{a}{b}\Delta y_v^2\right]} \le \sqrt{\frac{1}{2}[\Delta P_v^2 + \Delta P_h^2]} = \Delta P_v$$

It is seen that the selection $c = (a+b)/2$ resulted in $\Delta P_c \le \Delta P_v$. This shows that the perimeter of an ellipse can be approximated using a polygon of the perimeter of a circle of radius $c = (a+b)/2$.

$$\sum_i^n \Delta P_{c_i} \le \sum_i^n \Delta P_{v_i}$$

with

$\Delta p_{c_1} = \Delta p_{c_2} = \Delta p_{c_3} = \cdots = \Delta p_{c_n}$,

$\Delta p_{v_1} = \Delta p_{v_2} = \Delta p_{v_3} = \cdots \Delta p_{v_n}$, and

$\Delta p_{c_i} \le \Delta p_{v_i}$ for $i = 1$ dan $n \to \infty$

So, we get the perimeter of the circle (P_c) with radius $c = (a+b)/2$ less than or equal to the perimeter of the vertical ellipse (P_v) as follows:

$$P_c = 2\pi(a+b)/2 \le P_v \qquad (11)$$

where b and a are semi-major and semi-minor axes, respectively.

As we have seen, an elliptical curve can be partitioned into polygons. The length of the ellipse curve can be calculated by adding up each part of the polygon with the length of each part approaching zero $(\Delta P_{v_i} \to 0)$ or the number of partitions approaches infinity $(n \to \infty)$. If $\Delta P_{v_i} \to 0$ is in the interval $[p, q]$, then we can write equation (10) as follows

$$\lim_{n\to\infty} \textstyle\sum_i^n \Delta P_{v_i} = \lim\limits_{n\to\infty} \sum_i^n \Delta P_{c_i} = \lim\limits_{n\to\infty} \sum_i^n \sqrt{\Delta x_c^2 + \Delta y_c^2} \qquad (12)$$

If we write in the form of a parameter equation, i.e. $x_c = f(t), y_c = g(t), p \le t \le q$, and use the Mean Value Theorem for the derivative, we get the equation

$$\lim_{n\to\infty} \textstyle\sum_i^n \sqrt{\Delta x_c^2 + \Delta y_c^2} = \int_p^q \sqrt{[f'(t)]^2 + [g'(t)]}\ dt \qquad (13)$$

(J. Purcell & Varberg, 2016a). As we have seen, if $f(t) = c\cos t, g(t) = c\sin t, p \le t \le q$ then the result of equation (13) is the formula for the circumference of a circle.

Based on equations (10), (12), and (13), we can guarantee that the perimeter of the ellipse, i.e., $\pi\sqrt{2(a^2+b^2)}$, is an integral result of equation (13). Indirect proof using the integral in the equation of a circle can guarantee that it is not wrong.

Equation (9) proves that the distance between two points on an ellipse (ΔP_v), however large, will always remain the same distance when transformed into a circle of radius c. If the distance between these two points is part of the partition on an interval that approaches zero ($\Delta P_v \to 0$), then we make sure that the integral in equation (12) shows the circumference of the ellipse.

The results of this study confirm the research that has been done previously. The results of this study indicate that the ellipse is right at the upper bound, namely $2\pi\sqrt{(a^2+b^2)/2}$. The results of previous studies using Minkowski Sums by E. Pfiefer, Cauchy-Schwarz Inequality for integrals by Jameson, and parametric equations by Gusić show that the circumference of an ellipse has the upper bound $2\pi\sqrt{(a^2+b^2)/2}$ (E. Pfiefer, 1988; Gusić, 2015; G. J. O Jameson, 2015). Euler also obtained the same result through an approximation approach, without a series of hyper-geometric functions, that the perimeter of the ellipse is $2\pi\sqrt{(a^2+b^2)/2}$ (Almkvist & Berndt, 1988). The precision of Euler's approximation is more accurate if it includes a series of hyper-geometric functions. The results of other studies show that the maximum circumference of the ellipse is $4\sqrt{(a^2+b^2)}$(E. J. F. Primrose, 1973). This maximum circumference is slightly larger when compared to $2\pi\sqrt{(a^2+b^2)/2}$.

In this article, the author cannot compare the accuracy of the results of the calculation of the integral formula with what has been done by previous researchers. Previous researchers used the method of numerical approximation or integration. This method produces an elliptic integral study. The elliptic integral returns the result of a hyper-geometric function with infinite series. The results of the elliptic integral by some experts differ from each other by the degree of calculation error they have. Meanwhile, this article presents the formula for the Riemann integral (definite integral) on the ellipse, indirectly through the circle function as in equation (13).

CONCLUSIONS AND SUGGESTIONS

Transform the ellipse $(x^2/a^2 + y^2/b^2 = 1, a \geq b > 0)$ coordinates to circular coordinates $(x'^2 + y'^2 = c^2)$ with $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (c/a)x \\ (c/b)y \end{bmatrix}$ and $c = \sqrt{(a^2+b^2)/2}$ satisfy the condition to maintain the length of the polygon in the ellipse. When transforming an ellipse to a circle by maintaining the length of its inscribed polygon, then the approximation of the perimeter of the ellipse can use the approximation of the polygon in the circle. The perimeter of an ellipse based on a coordinate transformation that maintains the distance between its coordinates or polygons is $K(E) = 2\pi c = 2\pi\sqrt{(a^2 + b^2)/2}$. It is possible to derive the ellipse formula by transforming the ellipse into a circle with a radius $c = \sqrt{(a^2 + b^2)/2}$ that can be proven by using parametric equations for both circles and ellipses. Through parametric equations, we can prove that the integral for the perimeter of an ellipse is equal to the perimeter of a circle with radius *c*.

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